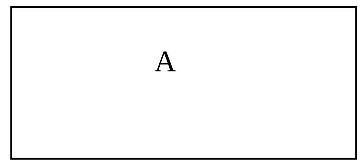
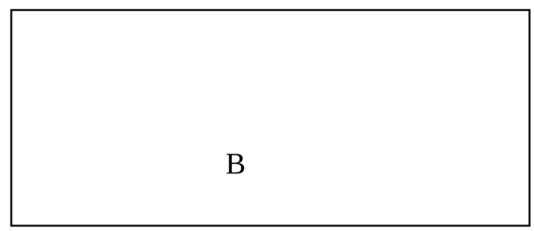
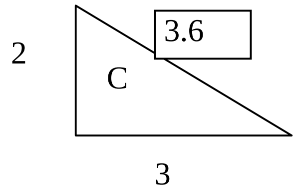
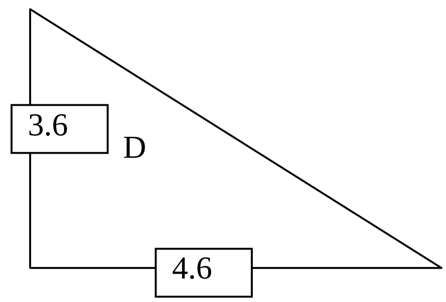
Vocabulary: ***Stretching and Shrinking***

Concept	Example
Similar: Two figures are similar if (1) the measures of their corresponding angles are equal *and* (2) the lengths of their corresponding sides are related by the same factor, called the scale factor. Similar figures have the same shape and are proportional copies of one another. Scale factor: To find the scale factor that relates two similar figures we must divide the length of one side of one figure by the corresponding length in the other figure. To decide which are corresponding lengths we need to pay attention to position in the figure. For example, in two triangles we might ask, “Are the two sides both opposite the largest angles?” If so then the sides are corresponding. This scale factor must be the same for every pair of corresponding lengths to make the figures similar.	*Are the following two rectangles similar?*  The sides of rectangle A are given as 2 units and 4 units.  The sides of rectangle B are given as 3 units and 6 units. Since all angles are right angles in both figures, we know that corresponding angles are equal. AND if we compare corresponding sides we have long edge of rectangle A: long edge of rectangle B = 4: 6, and short edge of rectangle A: short edge of rectangle B = 2:3. Since the ratios of corresponding sides are the same we can say that the rectangles are similar. The scale factor from rectangle A to rectangle B is $\frac{3}{2}$. If we multiply lengths in rectangle A by $\frac{3}{2}$ we get the corresponding lengths in rectangle B.

Are the following two right triangles similar?



Sides of right triangle C are given as 2, 3, 3.6 units. Only one angle is known, the right angle.

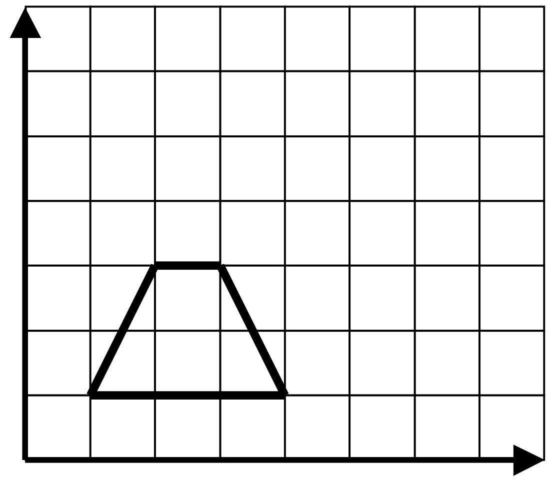


The above is also a right triangle. Since we do not have any information about other angles we can not determine if corresponding angles are equal or not. The legs (short sides) are given as 3.6 and 4.6. Can the triangles be similar?
Comparing the shortest legs we have 3.6:2, which is the same as 1.8:1. Comparing the longer legs we have 4.6:3, which is the same as 1.53:1. These are not equal ratios, so the triangles CAN not be similar.

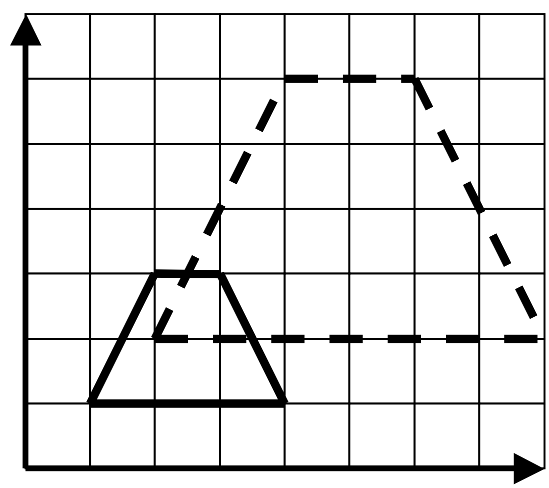
Similarity transformation:

Students scale up (or down) figures by using tools such as rubber bands, or by using rules applied to the coordinates of an original figure to transform the original figure into a similar figure. In general, if the coordinates of a figure are (*x, y*), algebraic rules of the form (*nx* + *a*, *ny* + *b*) will transform it into a similar figure with a *scale factor of n.* These algebraic rules are called *similarity transformations*. The addition of an *a* or a *b* does not change the scale; it would move the shape to another location (More on this in *Hubcaps and Kaleidoscopes.)*

Apply the rule (x, y)→ (2x, 2y) to the following figure. Compare the resulting figure to the original.



The vertices of the above figure are (1,1), (4,1), (3,3), (2,3). If we apply the transformations (x,y) → (2x, 2y) then a new set of points will be produced: (2,2), (8,2), (6,6), (4,6). This image is shown below, with the original. The angles are preserved, and corresponding sides are in the ratio 1:2. The scale factor is 2. The new figure is similar to the original.



Test for Similarity:

The safest test to apply to two figures to see if they are similar is to check that *all* corresponding angles are equal and that *all* corresponding sides are in the same ratio. (We actually may not need to match ALL angles and ALL sides to be sure that 2 figures are similar. However, the number of matched corresponding pairs of angles and/or sides sufficient to ensure similarity varies from one type of polygon to another.)

Similarity of Triangles:

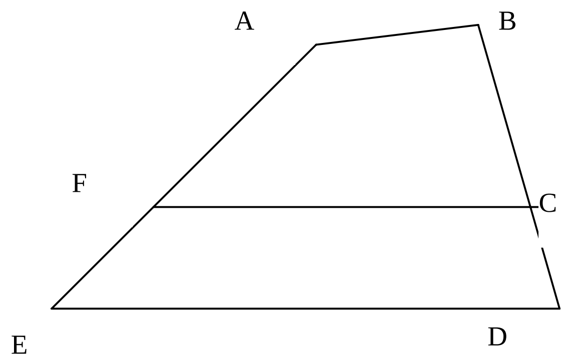
If the three angles in one triangle are equal to the three angles in another triangle then the triangles will be similar. It turns out that the corresponding sides will automatically be in the same ratio, even if this is not an explicit intention. Therefore, the size of the angles controls the shape (but not the size) of a triangle. (This is not true in general of a quadrilateral. That is, matching 4 angles in 1 quadrilateral with 4 angles in another quadrilateral will not ensure similarity.)

Note: If we know two angles in one triangle are equal to two angles in another triangle, then the third angles will also be equal, since the angle sum has to be the same (180 degrees) in each case. Thus, two triangles will be similar if we can show that *two* angles of one match two angles of the other.

Note: There are other conditions that ensure similarity of triangles, but these are not part of this unit. For example, if the three sides of one triangle form the same ratio with the three sides of another triangle the corresponding angles will automatically be equal without making that an explicit intention, and the triangles will be similar. More on these conditions in *Hubcaps and Kaleidoscopes*.

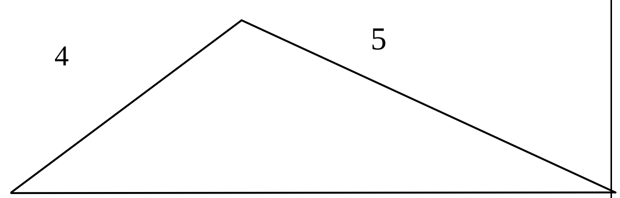
Alternative test for Similarity:

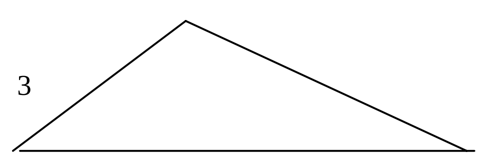
Are the two quadrilaterals shown below, ABCF and ABDE, similar, given FC is parallel to ED?



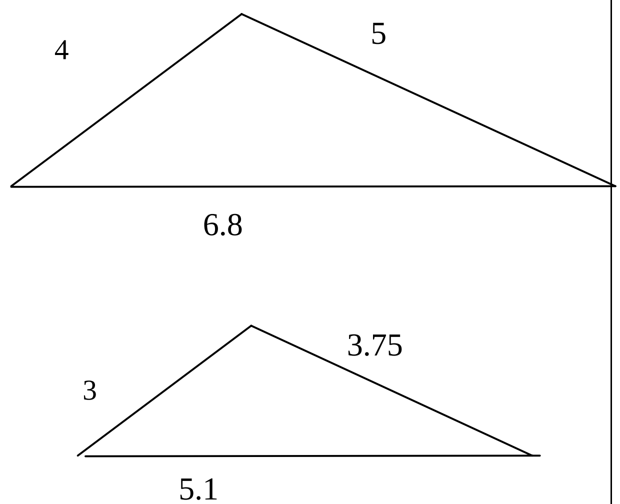
In the above figure FC is given parallel to ED. This ensures that angle AFC = angle AED, and angle BCF = angle BDE. Therefore, the four angles of quadrilateral ABCF are equal to the four corresponding angles of quadrilateral ABDE. However, we can see that AB in one quadrilateral corresponds to AB in the other quadrilateral, so the ratio of these sides is 1:1. Meanwhile, BC corresponds to BD, and the ratio is obviously not 1:1. Therefore, even though all corresponding angles match, the quadrilaterals are not similar.

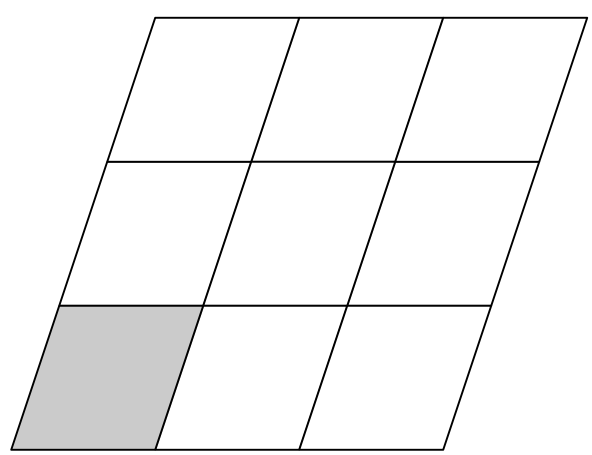
The corresponding angles are given equal in the triangles shown below. Are the two triangles similar? If so, what information can we deduce about side lengths?

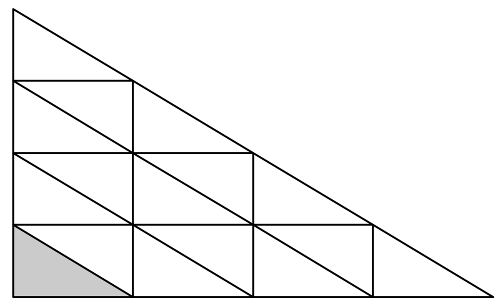


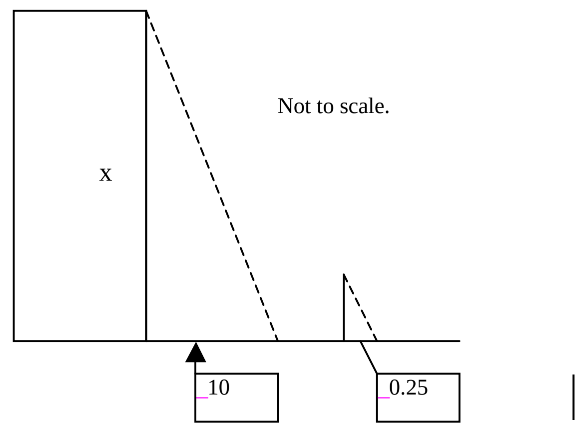


In the above triangles we are given that corresponding angles are equal. This, by itself, is

See below.	corresponding angles are equal. This, by itself, is enough to make the triangles similar. Therefore, we can **find the scale factor from the large triangle to the small triangle by comparing corresponding sides. The scale factor is $\frac{3}{4}$.** Because the triangles are similar this scale factor will transform any length on the large triangle to the corresponding length on the small triangle. Thus, we can find another side of the small triangle by applying the scale factor $\frac{3}{4}$ to the side labeled 5, to get $5(\frac{3}{4}) = 3.75$. We have no information about the third sides of the triangles.
Using Equivalent Ratios In the comparison of two similar figures, there are several equivalent ratios. • Some are formed by comparing lengths **within** a figure. This, with the existence of corresponding angles being equal, is an **alternative test for similarity.** • Others are formed by comparing lengths **between** two figures. This comparison is the **scale factor**.	*Are the two triangles shown below similar?*  We can show that the above triangles are similar two different ways: • *By comparing corresponding side lengths between* triangles we can show that the same scale factor transforms lengths of sides in the large triangle into lengths of corresponding sides in the small triangle. In fact the **scale factor** is $\frac{3}{4}$ or 0.75. $4(0.75) = 3$; $5(0.75) = 3.75$; $6.8(0.75) = 5.1$. This ensures similarity of the triangles. • *By comparing lengths within each* triangle we can show that **internal ratios** are the same. In fact, comparing the **shortest side to medium side in each triangle we**

	have $\frac{4}{5}$ and $\frac{3}{3.75}$, which are equal ratios. (Both are the same as $\frac{12}{15}$ or 0.8:1) Likewise, shortest to longest sides produces $\frac{4}{6.8}$ and $\frac{3}{5.1}$, which are equivalent ratios. And medium to longest lengths produces $\frac{5}{6.8}$ and $\frac{3.75}{5.1}$, again equivalent. Forming corresponding ratios **within** each figure produces the same ratio. This ensures similarity of the triangles.
Relationship of Area and Perimeter in Similar Figures: If two figures are similar with a **scale factor** of n, then each side of one figure is related to the corresponding side in the other figure by a scale factor of n, AND ***perimeters*** *of the two figures are also related by a scale factor of n*. That is, multiplying each side of one figure by a factor of n will produce the length of the corresponding side in the other figure, and multiplying the perimeter of one figure by a factor of n will produce the perimeter of the other figure. However, *the* ***areas*** *are related by a factor of* n^2; that is, multiplying the area of one figure by a factor of n^2 will produce the area of the other figure.	*Example with similar parallelograms:* The figure below is a large parallelogram made of copies of the smaller parallelogram (shaded). We can see that the angles of the small parallelogram are same as the angles of the large parallelogram because of the way that the large parallelogram has been formed. We can also see that the sides of the small parallelogram have been enlarged by a factor of 3 to make corresponding sides of the large parallelogram. The large parallelogram is similar to any of the small parallelograms. The **scale factor is 3**. Multiplying the perimeter of a small parallelogram by 3 produces the perimeter of the large parallelogram. However, the area of the large parallelogram is 9, **or** 3^2, times the area of the small parallelogram.  *Example with similar triangles:*

	The largest triangle below is made of copies of the small triangle (shaded). The large triangle is similar to any of the small triangles. The scale factor is 4. However, the area of the large triangle is 16 or 4^2 times the area of the small triangle.  Note: the triangle and parallelogram shown above are examples of **rep-tiles**, shapes which combine with copies of themselves to make a similar shape.
Finding missing lengths in similar figures: **Scale factor** can be used to solve for missing lengths. Alternatively **equivalent fractions** can be used to set up **proportions** and solve for missing parts.	*Example:* *Shadows made by the sun can be thought of as sides of similar triangles, because the sunlight hits the objects at the same angle. Shown below is a building of unknown height and a meter stick, both of which are casting shadows. Find the height of the building.* We can use the scale factor between the lengths of the shadows. Since enlarging from 0.25 to 10 involves a **scale factor of 40**, we multiply the height of the meter stick (1 meter) by 40 to obtain the height of the building, or 40 meters. We could also think of this as $\frac{x}{10} = \frac{1}{0.25}$ and use **equivalent fractions** to find the value of *x* that would make the ratios equivalent.


Not to scale.
x
10
0.25