Connected Mathematics 2

The Mathematics of the Number and Operations Strand

Prime Time – Grade 6

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The Mathematics of the Number and Operations Strand

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Summary of Investigation

Mathematics Background

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Summary of Investigation

Mathematics Background

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Content Connections

Connected Mathematics 2

The Mathematics of the Number and Operations Strand

Mathematics of the Unit

Pearson Prentice Hall Professional Development

Overview

Many important arithmetic problems involve breaking a whole number into equal-size pieces or finding a number into which a given number will divide evenly. Solving problems like these involves finding factors and multiples. For example:

- A class of 30 students is to be divided into equal-size teams for a school competition. What team sizes are possible?
- Frida and Georgia want to go to the art museum together the next time they both have a day off from work. Frida has a day off every fourth day. Georgia has a day off every fifth day. They both had the day off today. In how many days will they be able to go to the museum together?

Solving the first problem involves finding factor pairs of 30. The class can be divided into 1 team of 30, 2 teams of 15, 3 teams of 10, 5 teams of 6, 6 teams of 5, 10 teams of 3, 15 teams of 2, or 30 teams of 1. One of the most curious and important properties of the whole number system is that the answer to this question depends greatly on the number being divided. For example, if the class had just one more student, it could only be divided into 1 team of 31 or 31 teams of 1.

The second problem involves multiples. We need to find the least number of which both 4 and 5 are factors. This number is 20, the least common multiple of 4 and 5. Frida and Georgia can go to the museum together in 20 days.

Solving grouping and repeated-action problems like those above depends on finding factors and multiples of whole numbers. Realizing that some numbers are rich in factors, while other numbers have very few factors, is essential for effective problem solving. A primary goal of this unit is to help students learn some new and useful strategies for finding factors and multiples of whole numbers. They can then apply these strategies to gain familiarity with prime and composite numbers and to solve real-life problems.

Summary of Investigations

Investigation 1

Factors and Products

The Factor Game engages students in a friendly contest in which winning strategies involve distinguishing between numbers with many factors and numbers with few factors. Students are then guided through an analysis of game strategies and introduced to the definitions of *prime numbers* and *composite numbers*. The Applications—Connections—Extensions (ACE) questions are rich in connections to situations in which factors, multiples, and prime numbers are significant.

In the Product Game, students find products of factors. Although students develop strategies to win the game, the focus is on basic multiplication facts. Students then create their own games by selecting factors, determining products, and choosing appropriate dimensions for their game boards.

Investigation 2

Whole-Number Patterns and Relationships

Students make rectangles to represent models for numbers in Problem 2.1. In Question B of Problem 2.1, students use the rectangles they have created to consider how far they must go to be sure that they have found all the factors. You might use the language of the "turn around" point to describe this location where the factors we are finding in pairs give no new factors.

Problem 2.2 encourages conjecturing and creating arguments to support those conjectures. This problem is not essential if time is an issue. It does, however, put students in an environment that allows a conversation about mathematical argument and proof.

Students explore factors and multiples with Venn diagrams in Problem 2.3. The use of Venn diagrams pushes students to begin to notice important things about numbers and their factors and multiples. While Venn diagrams are not a good tool for finding factors and multiples, they are a very good representational device to focus students' attention on the common factors and common multiples of two numbers.

Investigation 3

Common Multiples and Common Factors

Real-life situations are used to motivate student interest in common factors and common multiples. The concepts of least common multiple and greatest common factor, though not formally introduced, are used naturally throughout the problems and in the ACE section. The context of the problems and questions helps make clear whether a solution involves finding a common multiple, a common factor, the least common multiple, or the greatest common factor.

Investigation 4

Factorizations: Searching for Factor Strings

Finding longer and longer factor strings of a number leads students to discover the Fundamental Theorem of Arithmetic: a whole number can be factored into a product of primes in exactly one way, disregarding order. The idea is to help students see that every string shorter than the longest has at least one factor that is not prime. These factors can be broken down further to make a longer string. The process ends when every number in the string is prime and no further breaking down can occur. When you reach this stage, you have the one and only string of primes that can be multiplied together to make the original number—thus arriving at the unique prime factorization of the original number. (Of course, the order in which the prime factors are listed is discounted.) The discussion of why 1 is not a prime number occurs in the ACE section.

Investigation 5

Putting It All Together

This problem is an option that gives students a chance to use a lot of what they have learned in the unit to solve something more challenging. If your time is limited, you might assign the locker problem as extra credit for interested students.

Mathematics Background

Factors and Multiples

Prime Time addresses the basics of number theory: factors, multiples, prime and composite numbers, even and odd numbers, square numbers, greatest common factors, and least common multiples. The concepts of factor and multiple are interdependent. If A is a factor of B, then B is a multiple of A. This means that we can find a number C such that the product of A and C equals B, that is, $A \times C = B$. From this we see that factors always come in pairs. For example, we know that $3 \times 4 = 12$. This says that 3 is a factor of 12 and that 4 is a factor of 12. The two are a factor pair because their product is equal to 12. In fact, there are several statements that can describe the relationships in the number fact $3 \times 4 = 12$. We can say the following:

3 is a factor of 12.
4 is a factor of 12.
3 is a divisor of 12.
4 is a divisor of 12.
12 is the product of 3 and 4.
12 is a multiple of 3.
12 is a multiple of 4.
12 is divisible by 3.
12 is divisible by 4.

It is important that students learn to use this language with meaning. We ask in a number of places for students to write such statements about a given number relationship.

Classifying Numbers by the Sum of the Proper Factors

An *abundant* number is one for which the sum of the proper factors of the number is greater than the number itself. The number 24, for example, is abundant because the sum of its *proper* factors is more than 24. A *deficient* number is one in which the sum of the proper factors is less than the number itself. The number 16 is deficient because the sum of its *proper* factors is less than 16. A perfect number is one in which the sum of the proper factors is equal to the number itself. The number 6 is *perfect* because the sum of its *proper* factors equals 6. Note that 6 and 28 are the only perfect numbers between 1 and 30.

The Multiplicative Identity

Students observe that 1 is a factor of every whole number. The product of 1 and another number, A, is A; that is, $1 \times A = A$. Hence we call the number 1 the *multiplicative identity.* For similar reasons, 0 is the *additive identity*. Zero plus any whole number equals the whole number; that is, $0 + A = A$. These ideas are useful when discussing multiplying and dividing fractions. These are important mathematical ideas that will be discussed in later CMP units.

Finding Near-Perfect Numbers

A near-perfect number is one whose proper factors sum to 1 less than the number itself. For example, the number 4, with proper factors 1 and 2, is near-perfect, because $1 + 2 = 3 = 4 - 1$. The number 16 is near-perfect because $1 + 2 + 4 + 8 = 15 = 16 - 1$.

Near-perfect numbers are useful for finding perfect numbers. Euclid discovered this method:

1. Start with a near-perfect number whose proper factors have a prime sum.
2. Multiply the sum of the factors by the greatest power of 2 less than the sum. The product will be a perfect number.

Examples: The number 4 is near-perfect, and the sum of its proper factors is 3, which is prime. The greatest power of 2 less than 3 is 2, and $3 \times 2 = 6$, which is perfect. The number 8 is also near-perfect, and the sum of its proper factor is 7, which is prime. The greatest power of 2 less than 7 is 4, and $7 \times 4 = 28$, which is perfect. Euclid's method will not work for the near-perfect number 16 because the sum of its proper factors is 15, which is not prime.

Euclid's method always produces even perfect numbers. No one knows whether there are any odd perfect numbers, but we do know that powers of 2 (e.g., 2, 4, 8, 16, 32, etc.) are always near-perfect numbers.

Formal Proofs About Even and Odd Numbers

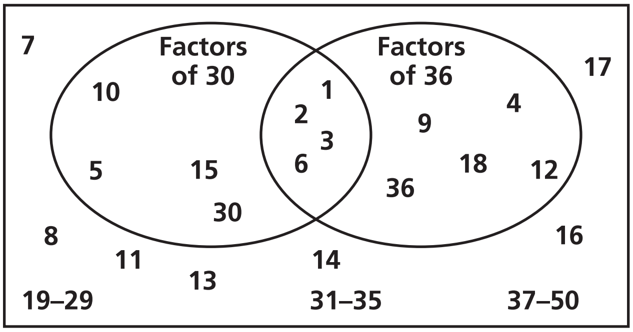
You might be interested in the formal proof that the sum of two even numbers is even. Any even number y can be written $y = 2n$. Let's take two numbers, $y = 2n$ and $z = 2m$. Then $y + z = 2m + 2n = 2(m + n)$, which is even! The proof that the sum of two odd numbers is even is similar: Any odd number a can be written $a = 2z + 1$. Let's take two odd numbers, $a = 2z + 1$ and $b = 2w + 1$. Then $a + b = 2z + 1 + 2w + 1 = 2z + 2w + 2 = 2(z + w + 1)$, which is even. The same sorts of proofs work for sums of even and odd numbers and the products of even and odd numbers.

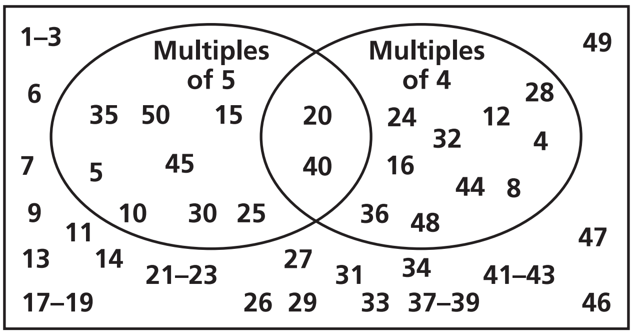
The Fundamental Theorem of Arithmetic

Through their work in these investigations, your students discover the Fundamental Theorem of Arithmetic. "Fundamental" theorems are few and far between in mathematics. This implies that the theorem is of "fundamental" importance to mathematics as a field and, in this case, especially to number theory. The Fundamental Theorem of Arithmetic states that every positive whole number can be written as the product of primes in exactly one way, disregarding order. For example, the number 120 can be written as $2 \times 2 \times 2 \times 3 \times 5$. Although you can switch the order of the factors—i.e., you can write $2 \times 3 \times 2 \times 5 \times 2$—every prime product string for 120 will have three 2s, one 3, and one 5. The Fundamental Theorem of Arithmetic helps us to see why 1 is not a prime number. In essence, the theorem states that a whole number can be identified uniquely by its prime factorization. That is, each whole number corresponds to a unique prime factorization, and each prime factorization corresponds to a unique whole number. If 1 were a prime number, this would not be true. Any string of primes could be extended with an unlimited number of 1s. We could say that the "prime" factorization of 12 is $3 \times 2 \times 2$ or $3 \times 2 \times 2 \times 1$ or $3 \times 2 \times 2 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$. From this, we see that we can express 12 as a product using as factors as many 1's as we like.

Common Factors and Common Multiples

We introduce Venn diagrams as a way of focusing students' attention on common factors or multiples of two numbers. The following are two of the Venn diagram problems in Investigation 2 with all numbers less than or equal to 50 placed in their appropriate place. We do not ask students to fill in all the numbers that fall outside the areas of the circles, but we do ask in some cases for them to fill in some that would fall outside.





As you can see, the Venn diagrams highlight the areas of overlap. In the first diagram you can see the common factors of the numbers. In the second diagram you can see some common multiples. This allows you to ask questions about what other numbers would be in the overlap if you kept checking numbers into the hundreds. In the case of common factors, no additional numbers would be added to the overlap (intersection). However, for common multiples there would be a never-ending stream of numbers we could add to the intersection—every multiple of 20 will be in the intersection.

The Least Common Multiple

If the two numbers have no common factors other than 1, then the least common multiple will be the product of the two numbers. ($13 \times 17 = 221$)

If the two numbers have a common factor, the least common multiple will be the product of the two numbers divided by the greatest common factor. ($12 \times 14 \div 2 = 84$, so 84 is the least common multiple.)

The Relationship of Factor Pairs to the Square Root of the Number

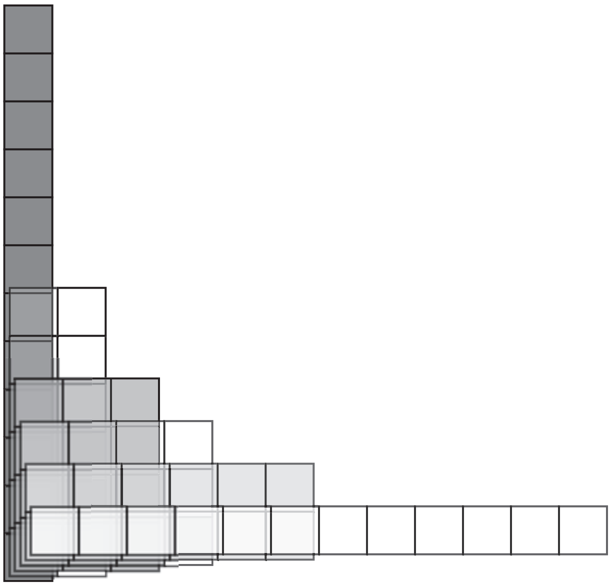
An important question arises naturally out of the investigation of rectangles one can make from a fixed number of tiles: How do I know when I have all the rectangles possible except for orientation? Another form of the question is: How do I find the "turn around" point in listing all the factor pairs of a number? A more sophisticated version of the question is: What numbers do I have to check to find all the factors of a number or to show that it is prime?

The key to finding all the factors of a given number, n, is to examine systematically the whole numbers that are less than or equal to the square root of n. Your students probably have little understanding of square roots at this stage, so they are more likely to see that the turn around occurs when the factors in the pairs get very close together. If we look at the factor pairs for 24 we see:

1, 24
2, 12
3, 8
Here, the factor pairs reverse. → 4, 6
6, 4
8, 3
12, 2
24, 1

Students begin to see that the 4×6 rectangle is the most square-like of the rectangles. They also notice that as one edge of a rectangle for 24 gets longer, the other edge gets shorter. This also implies that at some point the numbers in the column on the left get larger than the numbers in the column on the right. Where the order of size of edges changes is the turn around point.

Another way students have to see the turn around point is geometrically. By superimposing the rectangles for a number on top of each other in order, you can see the symmetry around the turn around point.

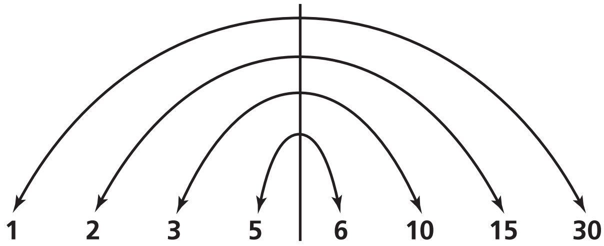


When the rectangles from factor pairs get as close to a square as possible, you have reached the turn around point and have found all the factors. When the number is a square number, you actually hit the turn around point with the rectangle that is a square. The length of the side of that square is the square root of the original number and the turn around point.

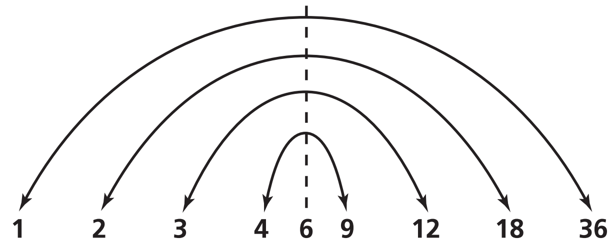
All the questions posed at the beginning depend ultimately on understanding that for any number, the two factors in a factor pair lie on opposite sides of the square root of the number. For example, analyze the factors and their pairings for 30 and 36.

The factors of 30 are:
1, 2, 3, 5, 6, 10, 15, 30.
The factors of 36 are:
1, 2, 3, 4, 6, 9, 12, 18, 36.

If we draw lines connecting the factor pairs in each, we get the following:



$\sqrt{30} = 5.48$



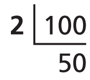
$\sqrt{36} = 6$

From the diagrams you can see that the square root becomes like a fulcrum around which factor pairs array themselves. We expect that your students will see that the pair of factors that make the most square-like rectangle indicates the place after which the factor pairs reverse and repeat themselves. Later in the unit when the students are finding the prime factorization of numbers, the answer to the question of how far one has to check to be sure a number is either prime or to find all of its prime factors is, once again, the square root. If you check all the prime numbers less than the square root, you will have found all the smaller prime factors of each factor pair involving a prime. If you find no such prime factors, then you can conclude that the number is prime.

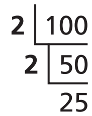
Finding Prime Factorizations

Another way to find the prime factorization of a number is to use a recording mechanism to help you keep track of the prime factors you have already found. In the Student Edition we suggest the following recording scheme with the example of the prime factorization of 100 given:

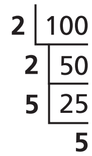
First, find one prime factor of 100. Start with 2, the least prime that divides 100. Divide 100 by 2, showing the work as an upside-down division problem.



Next, find a prime factor of 50. You can use 2 again. Add another "step" to the division problem.

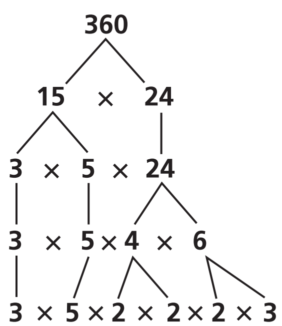


Now, find a prime factor of 25. The only possibility is 5. Add a third "step" to the division problem.



You are left with a prime number, 5. From the final diagram, you can read the prime factorization of 100: $100 = 2 \times 2 \times 5 \times 5$.

In the Teacher's Guide, we give another way to find the longest factorization of a number. This method is to make a factor tree. This method is useful because it suggests "breaking apart" numbers into their factor pairs and subsequently "breaking apart" the factors. Below we show this method for 360.



Students might develop a similar method on their own. If some students are having difficulty finding the longest (prime) factorization of a number, you might want to lead them to this method. The methods are equally valid, so students should use the one that makes the most sense to them.

Classifying the Number 1

The question of how to classify and use the number 1 arises in many circumstances. Mathematicians want to write whole numbers in terms of a *unique* factorization. If 1 were a prime number, we could make any string longer by multiplying by 1. The number 14 could be written 7×2 or $7 \times 2 \times 1$ or $7 \times 2 \times 1 \times 1 \times 1$. In addition, calling 1 a prime number violates the definition of "prime number." The Fundamental Theorem of Arithmetic is a theorem about factorization into primes, so we don't want to consider the number 1 as a prime. We cannot consider it composite, as it is not the product of two or more different whole numbers. Thus the number 1 is in a classification by itself. It is called the "unit" and is neither prime nor composite.

Finding Greater Prime Numbers

We know that the prime numbers grow sparser and sparser as whole numbers get greater, but is it possible that at some point prime numbers just "run out"? The answer is no: suppose there were a "greatest" prime number. Let's say there were k prime numbers in total. Suppose we label all of the primes: $n_1, n_2, n_3, \ldots$, and so on. So we would say $n_1 = 2, n_2 = 3, n_3 = 5, n_4 = 7$. We would eventually get up to n_k. Now let's look at the product of all of those primes: $n_1 \times n_2 \times n_3 \times \ldots \times n_k$. Of course this number isn't prime. But what happens if we add 1? Let's look at $n_1 \times n_2 \times n_3 \times \ldots \times n_k + 1$. Whatever prime number we choose can't divide *this* number evenly because the remainder when we divide will always be 1! And remember, we've supposed that *every* prime is included in this list. So, this number is prime! But it must be greater than n_k, which means that our original supposition that there is a "greatest prime" is wrong. There cannot be a greatest prime.

Large prime numbers are important in coding systems for transmitting secret information. The Electronic Frontier Foundation awarded Nayan Hajratwala of Plymouth, Michigan, $50,000 for finding a prime number with more than 2 million digits! He found the prime number in 1999 on his personal computer. He let the computer look for the prime number during idle time, and it took 111 days. We would write down the number for you, but in this type, it would be more than 2 miles long. The recent work (August 2002) by

Dr. Manindra Agrawal and two college students, Neeraj Kayal and Nitin Saxena, has made finding large primes much easier. A 7,235,733-digit prime number was found in May 2004.

The book *The Mathematical Tourist* by Ivars Peterson has a fascinating chapter called "Prime Pursuits" that might interest students.

Using Prime Factorizations to Find the Greatest Common Factor and the Least Common Multiple

Once you have the prime factorizations you can use them to find the greatest common factor (GCF) and the least common multiple (LCM). Here is an example:

$$72 = 2 \times 2 \times 2 \times 3 \times 3$$

$$120 = 2 \times 2 \times 2 \times 3 \times 5$$

The prime factors that the two numbers have in common are 2, 2, 2, and 3. To find the greatest common factor we can multiply these numbers to get $2 \times 2 \times 2 \times 3 = 24$. Thus, 24 divides each of the original numbers and no larger number does so.

The least common multiple can be found by taking the union of all the prime factors. This means that you take the part in common and multiply that by each of the prime factors that are not in common. So the LCM of 72 and 120 is $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$.

We do ask the students to find some numbers whose least common multiple is the same as the product of the two numbers. Examples are 72 and 35. If you look at the prime factorization of these two, they have no primes in common. Such numbers are **relatively prime**. They have no common factors other than 1. In such a case, the least common multiple is the product of the two numbers, in this case $72 \times 35 = 2{,}520$. If you used the same strategy as above, you would have multiplied the prime factorization of 72 by the prime factorization of 35. This is $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2{,}520$.

The Relationship Between the Greatest Common Factor and the Least Common Multiple

You can analyze the makeup of the least common multiple in another way.

The product of 72 and 120 is: $2 \times 2 \times 2 \times 3 \times 3 \times 2 \times 2 \times 2 \times 3 \times 5 = 8{,}640$.

You can see each of the prime factorizations of the two numbers in this product.

$$72 \times 120 = 8{,}640$$

$$(2 \times 2 \times 2 \times 3 \times 3) \times (2 \times 2 \times 2 \times 3 \times 5) = 8{,}640$$

Circle the GCF and the LCM of the two numbers.

GCF = 24 LCM = 360

$$(2 \times 2 \times 2 \times 3) \times (3 \times 2 \times 2 \times 2 \times 3 \times 5) = 8{,}640$$

So, $\text{GCF}(72, 120) \times \text{LCM}(72, 120) = 72 \times 120$, or $24 \times 360 = 8{,}640$.

We can solve this equation to find a rule for finding the LCM.

$$\text{GCF}(72, 120) \times \text{LCM}(72, 120) = 72 \times 120$$

$$\text{LCM}(72, 120) = \frac{72 \times 120}{\text{GCF}(72, 120)}$$

In general, $\text{LCM}(a, b) = \frac{a \times b}{\text{GCF}(a, b)}$.

Applied Problems

Another feature of this unit is the set of applied problems that engage the students in using their knowledge of primes, factors, multiples, factor pairs, and square numbers. These problems also create situations where the students have to figure out which of the several things they have learned to do is appropriate to help solve the problem. Gaining experience in selecting what is needed from among one's mathematical tools is critical for students if we expect them to be able to make use of what they know. Investigations 3 and 5 are devoted to creating such experiences.

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Content Connections to Other Units

Big Idea	Prior Work	Future Work
Determining the factors of whole numbers; finding the greatest common factor of two numbers	Learning and applying multiplication and division facts; applying the division algorithm (elementary school)	Performing arithmetic operations with fractions (*Bits and Pieces I, II & III*); comparing, scaling, and testing for similarity (*Stretching and Shrinking, Comparing and Scaling*); factoring algebraic expressions (*Frogs, Fleas, and Painted Cubes; Say It With Symbols; The Shapes of Algebra*)
Generating multiples of numbers; finding the least common multiple of two numbers	Learning and applying multiplication facts; counting by 2's, 3's, etc. (elementary school)	Understanding decimal numbers and the concept of place value (*Bits and Pieces I, II, & III; Comparing and Scaling*); identifying and analyzing patterns in the products of two numbers (*Covering and Surrounding; Variables and Patterns; Moving Straight Ahead; The Shapes of Algebra*)
Determining factorizations, including the prime factorization, of a whole number	Learning and applying multiplication and division facts; testing numbers for divisibility (elementary school)	Finding the LCM in order to find common denominators for fractions and ratios (*Bits and Pieces I, II, & III; Comparing and Scaling*); studying patterns in multiplicative relationships to develop algorithms for finding area, surface area, and volume of figures (*Covering and Surrounding*); identifying irrational numbers (*Looking for Pythagoras*); studying exponential relationships (*Growing, Growing, Growing*); developing and applying counting strategies (*Clever Counting © 2004*)
Classifying numbers as prime or composite, as even or odd, and as abundant, deficient, or perfect	Applying multiplication, addition, and division facts; comparing positive whole numbers (elementary school)	Classifying numbers as positive or negative (*Accentuate the Negative*) and as rational or irrational (*Looking for Pythagoras*); classifying relationships as linear, quadratic, inverse, or exponential (*Variables and Patterns; Moving Straight Ahead; Thinking with Mathematical Models; Growing, Growing, Growing; The Shapes of Algebra*)

Overview

Rational numbers are at the heart of the middle-grades experience with number concepts, but the concepts of fractions, decimals, and percents are often difficult for students. Research tells us that part of the reason for students' confusion about rational numbers is a consequence of the rush to symbol manipulation with fractions and decimals. A second reason fractions are difficult is that there are many useful interpretations of fractions and of the four basic operations used with fractions. Students need time to develop an understanding of and skill in using fractions and decimals and we have constructed *Bits and Pieces I, II, and III* and other number units in grade six to allow for a thoughtful, development over time for the students. The investigations in *Bits and Pieces I* are designed to help students to make sense of fractions, decimals, and percents in contexts that stimulate different models and interpretations of fractions.

The many different and important interpretations of and models for rational numbers can make grasping ideas about such numbers difficult. To gain a mature knowledge of rational numbers, students must be able to handle these various interpretations. We have carefully chosen the interpretations and models used in the unit. Some models are more powerful than others, as they contribute to developing the meaning of rational numbers and to understanding operations on rational numbers.

This unit helps the teacher create a supportive environment for students to grapple with interesting problems in which ideas of fractions, decimals, and percents are imbedded. As students work—individually, in groups, and as a class—they will develop productive ways of thinking about rational numbers. The teacher's role is to help students make explicit their growing ideas about rational numbers. In *Bits and Pieces II* we focus on operations with fractions and create an opportunity for students together with their teacher to develop algorithms for each of the basic operations. In the last number unit, *Bits and Pieces III,* we develop algorithms for operations on decimals as well as basic data analysis tools and strategies for solving problems involving percents. The fact that data analysis and percents make use of decimals gives contexts and payoff for the work on decimals.

Summary of Investigations

The goal of *Bits and Pieces I* is to help students make sense of fractions, decimals, and percents and to see the relationships among these three forms. Many students have had exposure to the area model in their elementary mathematics coursework. In order to expand the ways students can reason about rational numbers, this model is not used with fractions and percents. Rather, students will work with linear models such as fraction strips, percent bars and number lines. The area model is used in ACE problems and to help students understand decimals and place value.

Investigation 1

Fundraising Fractions

Students explore three components of understanding fractions: the visual model (fraction strips), word names for fractions, and symbols for fractions. Students attend to patterns and relationships between fraction representations and quantities while they are folding fraction strips. Through the fraction strips, the part-whole interpretation of fractions is developed. The measuring of progress in a school fundraiser focuses students on interpreting fund-raising thermometers as a representation of a fraction amount of a whole and as a fraction part of a monetary goal.

Investigation 2

Sharing and Comparing With Fractions

Investigation 2 develops equivalence, ordering and comparing fractions, and naming fractions greater than one. The focus is on the act of partitioning into equal-size pieces, and then repartitioning and renaming smaller and smaller parts. Candy licorice lace is used as a proxy for a number line. Four students going on a hike partially cut the licorice lace so it is easy to break, but have to re-mark the lace as more and more students join them for the hike. The new marks have to incorporate the old marks in order to name the fraction of the licorice lace each student will receive. In the process, students learn to recognize and generate equivalent names for a length.

By using equivalence, students develop strategies for finding a fraction between two given fractions. Students work with representing fractions greater than one.

Investigation 3

Moving Between Fractions and Decimals

Decimal-fraction relationships for halves, thirds, fourths, fifths, sixths, eighths, and tenths are developed across the first three problems. Once these decimal-fraction relationships are established students are asked to use them to develop strategies for finding decimals for other fractions.

In addition to developing decimal-fraction benchmarks, the first three problems in Investigation 3 use area models (square grids) and linear models (number lines and fraction strips) to introduce students to the decimal place-value system. In the second problem students investigate the continued subdivision of a 100-square grid to show 1,000 parts or 10,000 parts. This process of subdividing and naming the new parts is very important mathematically, as is developing strategies to find a decimal that falls between two given decimals. The notion of infinite or repeating decimals is only introduced in *Bits and Pieces I*. Repeating decimals will be explored in-depth in the third rational number text, *Bits and Pieces III*. Students also learn that the decimal place-value system is a way to interpret, compare, and order decimal numbers.

Investigation 4

Working With Percents

The percent bar model is used as a strategy for understanding what percent a fraction in a context represents. Students consider percents as a way of making comparisons among middle school basketball players' free-throw shooting data. Students learn how to write a percent for situations that are not based on 100 and develop strategies for changing forms of representation among fractions, decimals, and percents.

Mathematics Background

In this unit, students will meet several interpretations and models of fractions. The problems and their sequencing have been carefully chosen so that the move between problems will add to a deepening knowledge and comfort with fractions.

Interpreting the Notation for a Fraction

The terms *numerator* and *denominator* are introduced in and developed across Investigation 1. Students are introduced to the terms *numerator* as the number above the bar and *denominator* as the number below the bar. They are asked to think about what the numerators and denominators of fractions are referring to in contextual situations. The ultimate goal is for students to understand the role of each. The denominator of a fraction refers to how many parts of equal size are in the whole. The denominator also tells us the size of the parts in relation to other fractions with the same size whole. For example, students should be able to reason that a fourths piece is larger than a fifths piece because the whole is partitioned into less parts—making the size larger. The numerator refers to how many of the parts are of interest or are being referred to.

Look for opportunities throughout the unit to push students to use part-to-whole reasoning and to focus on the role of the numerator and the denominator. Help students to think about situations where it is useful to interpret the denominator as referring to the number of parts in the whole and those where it is helpful to interpret the denominator as referring to the size of a part. This is a subtle distinction, but it is helpful in solving problems involving fractions. It is important for students to be able to think about each of these roles when they are exploring fraction computation in *Bits and Pieces II.*

Mixed number is a useful term which applies to a number comprised of a whole number and a fraction. For example, $3\frac{1}{2}$ is a mixed number. *Improper fraction*, however, is less clear-cut but it may be required in your school for testing or standards. There is, in fact, nothing improper about a fraction with a numerator equal to or greater than the denominator. There is not even a mathematically useful distinction between fractions and improper fractions. It is probably best to simply treat them the same. The important thing for students to understand is that when the numerator is equal to or greater than the denominator, the value of the fraction is one or greater than one.

Interpretations of Fractions

The major interpretations on which this unit focuses are

- fractions as parts of a whole
- fractions as measures or quantities
- fractions as indicated division
- fractions as decimals
- fractions as percents

There are two major interpretations of fractions developed over *Bits and Pieces I, II,* and *III* and continued in the grade 7 units *Stretching and Shrinking* and *Comparing and Scaling*. The first is of fractions as *numbers*. We can add $\frac{1}{2}$, multiply by $\frac{1}{2}$, compare it to other numbers, and place it on the number line. Another interpretation is of fractions as *ratios* or related ideas such as fractions as operators ("stretchers" or "shrinkers") and fractions as rates or parts of a proportion. An example is when we say that $\frac{1}{6}$ of a school is sixth-graders. Strictly speaking, this is not a number but a ratio. Out of every six students, one is a sixth-grader. Because the whole is not specified, we do not know if there are more sixth-graders in this school than another where $\frac{1}{5}$ are sixth-graders. This is a situation where we often use percents. Percents express ratios.

Fractions as Parts of a Whole

This interpretation of rational numbers is applied in situations that are continuous and in situations that consider discrete objects. The important characteristic is that this interpretation depends on partitioning an object or a set of objects into equal-size parts or groups and making a comparison of some of the parts to the whole object or set. For example, if there are 27 students in the class and 13 are girls, the part of the whole that is girls can be represented as $\frac{13}{27}$. Or, if you have 75 candies for 5 kids to share, the fraction of the candies each will get can be represented as $\frac{1}{5}$. The whole or 75 is partitioned into 5 equal parts of 15. This can also be notated as $\frac{1}{5}$ of $75 = 15$.

In the following diagram, two parts are shaded.

The shaded portion can be represented as $\frac{2}{3}$. The 3 tells into how many equal-size parts the whole has been divided, and the 2 tells how many of the equal-size parts have been shaded.

In the part-whole interpretation of fractions, the difficulties for students center on the following:

- determining what the whole is
- subdividing the whole into equal-size parts—not necessarily equal shape, but equal size
- recognizing how many parts are needed to represent the situation
- forming the fraction by placing the parts needed over the number of parts into which the whole has been divided

Fractions as Measures or Quantities

In this interpretation, a fraction is thought of as a number. For example, a fraction can be a measurement that is "in between" two whole measures. Students meet this every day in such references as $2\frac{1}{2}$ brownies, 11.5 million people, or $7\frac{3}{4}$ inches. Understanding this interpretation is important for students' mathematical development, and it leads to comparison and ordering of fractions and operations on fractions.

Fractions as Indicated Divisions

To move with flexibility between fraction and decimal representations of rational numbers, students need to understand that fractions can be thought of as indicated divisions. Sharing is a natural context in which to help students see how this interpretation is related to whole-number division. If students see that sharing 36 apples among 6 people calls for division (36 apples ÷ 6 people = 6 apples each), then they can move to an understanding that sharing 3 apples among 8 people calls for dividing 3 by 8 to find out how many each person receives ($\frac{3}{8}$ of an apple).

Fractions as Decimals

A byproduct of the division interpretation of fractions is the relationship between a fraction and decimal representation of the same quantity. For the fraction $\frac{2}{5}$, for example, we can find the decimal representation by dividing 2 by 5. Given the modern tools of calculators and computers, decimal representations are even more important today than in the past. Students need time to develop comfort and ease in moving between fractions and decimals, and they need to understand decimals in two ways:

- as special fractions with denominators that are powers of 10
- as a natural extension of the place-value system for representing quantities less than 1

Fractions as Percents

Rather than treating fractions, decimals, and percents as separate topics, this unit seeks to build the connections among them. Students will see that the ideas and concepts are related and that the differences are in the symbols used to represent those ideas. Ten percent, 10%, is simply another way to represent 0.10 or 0.1, which is another way to represent $\frac{10}{100}$ or $\frac{1}{10}$. Percents are introduced as special names for parts of 100.

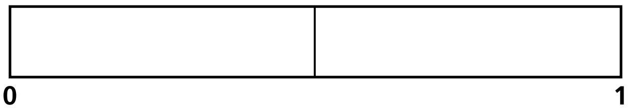
Models of Fractions, Decimals, and Percents

The models of rational numbers used throughout this unit were chosen because they connect directly to the interpretations of rational numbers explored in the unit. The models on which this unit focuses are

- fraction-strip models
- number-line models
- partition models
- grid-area models
- percent bar models

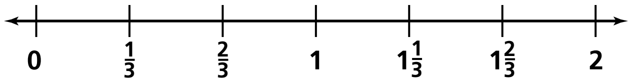
Fraction-Strip Models

Students are introduced to fractions in a situation in which use of a fraction strip as a model helps solve the questions posed. Fraction strips can be created by dividing a strip of paper into equal-size parts by folding. This is a fraction strip for halves:



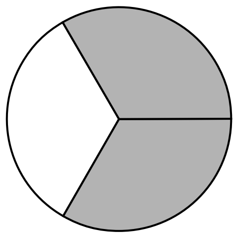
Number-Line Models

Fraction strips are used to motivate a number-line model of rational numbers. The number-line model helps make the connection to fractions as numbers and quantities. It also supports students' understanding of partitioning into and naming smaller and smaller parts. This is a number line from 0 to 2 with a few fraction quantities marked:



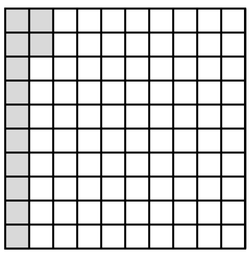
Partition Models

Students also use a more general model of fraction situations that is based on partitioning an area, such as a circle, or a length into equal-size parts. In the diagram to the right, the circle model to illustrate $\frac{2}{3}$ shows a partition of an area. The diagram below shows a partition of a length on a number line or ruler.



Grid-Area Models

Because 100 and powers of 10 are so useful in understanding fraction and decimal relationships, grid-area models are introduced and developed in this unit. This grid shows a shaded area of $\frac{12}{100}$ or 0.12.

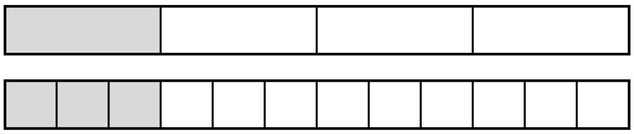


Percent Bar Models

A percent bar uses the idea of partitioning a length, as on a number line, but in this case two different number lines are related—one to show percents and one to show partitions of the amount that needs to be represented as a percent. An example is given below (Figure 1). The example shows that 84 is equivalent to 70% of 120.

Equivalence of Fractions

Partitioning and then partitioning again is an important skill that contributes to understanding equivalence of fractions. For example, if a licorice lace is marked into fourths (the first partition) and then each fourth is marked into thirds (the second partition), each original fourth has three parts (or three twelfths) in it. Thus one fourth is equivalent to three twelfths. Because each fourth was marked into three parts, the size of the parts (denominator) is one third of the original fourth (which is $\frac{1}{12}$) and the number in the whole (denominator) tripled. It will take three "times" as many of the new pieces (numerator) to create the original one fourth.



$\frac{1}{4} = \frac{3}{12}$

It is no accident that one multiplies by three when moving from fourths to twelfths numerically. It is because of the way the original partition is repartitioned. In the case when we start with $\frac{3}{12}$ and move to $\frac{1}{4}$, it involves ways to regroup the smaller partitions or units to create larger ones. As students look for patterns in their partitioning, help them to visually see the relationship developed between subsequent partitions. Students will apply this idea in the section where the algorithm is formally developed.

Fraction Benchmarks

Benchmarks are numbers that are easy referents such as 0, $\frac{1}{2}$, and 1. One way to estimate the size of a fraction is to compare it to these benchmarks. The set of benchmarks is expanded as students work through the unit to include $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}$ in fraction, decimal, and percent form. The point of this work on benchmarks is to develop a strategy for estimating relative size of fractions whether in fraction, decimal, or percent form. In developing operations in *Bits and Pieces II* and *Bits and Pieces III,* these ideas become a key to estimating sums and products. Students need to know this list in all forms to become good estimators.

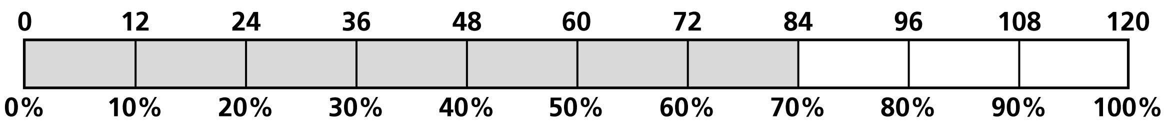
Fractions Between Fractions

Being distributed on a number line so that between any two fractions there is another fraction makes fractions quite useful in measurement contexts because the partitioning allows finer and finer grained measures.

We say the rational numbers are "dense" in the set of real numbers, meaning that within each interval on the real line, we can find a rational number. In particular, between any two irrational numbers, we can find a rational number. This might not seem very surprising. After all, both sets are infinite. However, the rational numbers are "countable," meaning that we find a one-to-one correspondence between the rational numbers and the natural numbers. As a set, the rational numbers are the same "size" as the set {1, 2, 3, 4, . . .}. The irrational numbers are not countable—the set of irrational numbers is much "bigger" than the set of rational numbers. So much bigger, in fact, that the probability of drawing a rational number from a theoretical bag containing all the real numbers is 0. So it is surprising that a set as "small" as the rational numbers would be "dense" in the real numbers.

Two good books to read to learn more are *What is Mathematics?* by Richard Courant and Herbert Robbins and *Journey Through Genius* by William Dunham.

Figure 1

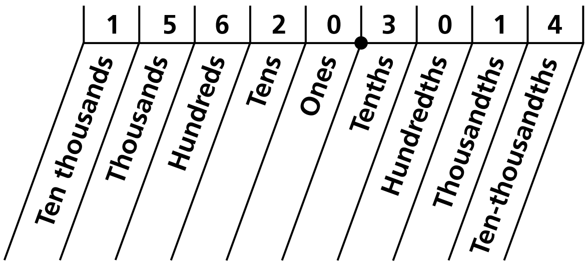


Place Value Notation for Decimal Fractions

Fraction notation makes clear to the reader the number of units and the size of the unit being used. For example with the fraction $\frac{4}{5}$ the denominator indicates that the size of the unit is fifths. The numerator indicates that 4 fifths are being used or referenced. The same can be said of the fraction $\frac{8}{10}$. By simply saying the fraction, we know that we are working with tenths and we have 8 of them. This is not the case with base-ten notation. When using base-ten notation to write numbers, the size of the unit is implied by the placement of the digit. For example, we know that 0.8 is read "eight tenths" and represents eight tenths because we know that a digit in the place to the right of the decimal is a tenth. The digit in this place-value spot represents the number of fractional parts of one whole when the whole is partitioned into 10 parts.

The system we use for writing numbers is called the base-ten number system. It uses groups of ones, tens, hundreds, thousands, ten thousands, and so on. For example, the whole number 69 represents 6 groups of ten and 9 groups of one; 28,590 represents 2 groups of ten thousand, 8 groups of one thousand, 5 groups of one hundred, 9 groups of ten, and 0 groups of one.

Over time, people realized they needed to extend the number system to represent numbers smaller than 1. A decimal point separates these digits so that the numbers to the right of the decimal point represent fractions whose denominators are ten (tenths), one hundred (hundredths), one thousand (thousandths), ten thousand (ten-thousandths), and so on. For example, 5.8 represents 5 groups of one and 8 groups of one tenth; 36.420 represents 3 groups of ten, 6 groups of one, 4 groups of one tenth, 2 groups of one hundredth, and 0 groups of one thousandth. Here is a place-value chart for 15,620.3014.



Ordering Decimals

One advantage of decimals over fractions is ease of ordering. In general, to compare the size of two decimals, we can compare them one decimal place at a time. If we start from the largest decimal place (furthest to the left), we need only look for the first place where the two numbers differ. The number with the largest digit in that place is larger. For example, 2,002.13 is larger than 2,002.016 because 2,002.13 has a 1 in the tenths place, while 2,002.016 has a 0 in the same place. Ordering decimals does not require the same kind of creative and flexible thinking that ordering fractions does. **Yet, decimals violate the first ordering principle students may have come to understand about whole numbers—the whole number with more digits is larger. Students logically want to use this rule for comparing decimals.** Students will need support in sorting this out.

Reference Resources for the Unit

Courant, Richard and Herbert Robbins. *What is Mathematics?* 2nd ed, revised by Ian Stewart. NY: Oxford University Press, Inc: 1996.

Dunham, William *Journey Through Genius*, New York: Penguin Books, 1991.

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Content Connections to Other Units

Big Idea	Prior Work	Future Work
Understanding, comparing, and applying fractions	Comparing whole numbers and finding the least common multiple (*Prime Time*)	Developing algorithms for performing calculations with fractions (*Bits and Pieces II*); developing algorithms and performing calculations with decimals (*Bits and Pieces III*); using scale factors (*Stretching and Shrinking*); applying rational numbers (*Covering and Surrounding; Comparing and Scaling*); interpreting slope (*Moving Straight Ahead*); interpreting fractions as probabilities (*How Likely Is It?; What Do You Expect?*); identifying and finding equivalent expressions (*Say It With Symbols*)
Understanding, comparing, and applying decimals	Comparing whole numbers; exploring multiples of 10 (*Prime Time*)	Developing algorithms and performing calculations with decimals (*Bits and Pieces III*); interpreting decimals as probabilities (*How Likely Is It?; What Do You Expect?*); applying rational numbers (*Bits and Pieces II; Covering and Surrounding; Bits and Pieces III; Comparing and Scaling; Samples and Populations*); using decimals to express, compare, and work with very large or very small numbers (*Data Around Us* ©2004)
Understanding, comparing, and applying percents	Comparing whole numbers; finding the greatest and least common multiple of two whole numbers (*Prime Time*)	Applying rational numbers (*Bits and Pieces II; Comparing and Scaling; Samples and Populations*); interpreting percents as probabilities (*How Likely Is It?; What Do You Expect?*); working with statistics and data reported as percents (*Data Around Us ©2004*)
Connecting fractions, decimals, and percents	Studying multiples (*Prime Time*)	Using fractions, decimals, and percents as expressions of probabilities (*How Likely Is It?; What Do You Expect?; Samples and Populations*); using fractions and decimals as slope or variable coefficients in equations (*Variables and Patterns; Moving Straight Ahead; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols; Thinking With Mathematical Models*); connecting fractions, decimals, and percents by interpreting percentages as decimals and fractions (*Bits and Pieces II; Bits and Pieces III; Comparing and Scaling*)

Overview

The overall goal of *Bits and Pieces II* is to develop meaning for and skill with computations involving fractions. When students finish this unit they should know algorithms for computations that they understand and can use with ease. This unit does not teach a specific or preferred algorithm for working with rational numbers. Instead it helps the teacher create a classroom environment in which students work on problems and generate ideas and strategies that make sense to them. At a point in the development of each operation, students are asked to pull together their strategies into an algorithm that works for all fraction situations involving that operation. As they work individually, in groups, and as a whole class on the problems, they develop and practice the algorithms to develop skill in carrying them out. This development process allows students to recognize special cases that can be easily handled and yet ends with students having an efficient general algorithm that works for all cases within an operation.

Letting the students wrestle with making sense of situations may take more time in the beginning. However, the payoff in the long run is that students learn to think and to reason about mathematical situations and although the algorithms need practice, they will not need to be taught repeatedly. The invented algorithms of students are often efficient and can evolve into standard algorithms. As they do, students understand why standard algorithms work.

We expect that when students finish this unit they will have an understanding of the meaning for computations with fractions. Students should be able to decide which operation is appropriate in a given situation. In addition, students should be able to use number sense, benchmarks, and operation sense to estimate solutions for computational situations as well as use estimation to decide if exact answers are reasonable.

Summary of Investigations

Investigation 1

Estimating With Fractions

Investigation 1 focuses on estimating sums of fractions and decimals. It builds on work with benchmarks, ordering, and the relationship between decimals and fractions in *Bits and Pieces I*. Students play a game in which they estimate the size of sums. Students also explore underestimation and overestimation as a strategy to reason about contextual situations where an estimate is needed and it is important to know if the estimate is above or below an exact amount.

Investigation 2

Adding and Subtracting Fractions

This investigation focuses on developing computational understanding and skill in adding and subtracting fractions. This investigation does not *give* students algorithms for computation. Instead, it prepares students to figure out how to add and subtract fractions by emphasizing flexibility in finding equivalent fractions. In the course of solving the problems, students develop strategies for adding and subtracting fractions and mixed numbers. Through class discussion these strategies are made more explicit and efficient. The inverse relationship between addition and subtraction is developed through the exploration of fact families. Although many students understand that addition and subtraction are related in whole-number contexts, they do not always extend this idea to include fractions. The last problem gives students presorted addition and subtraction problems to solve, characterize, and from which to create a general algorithm.

Investigation 3

Multiplying With Fractions

Investigation 3 focuses on developing computational skill with and understanding of fraction multiplication. Various contexts and models are introduced to help students make sense of when multiplication is appropriate. The first two problems develop multiplication with simple fractions and the third and fourth problems focus on fraction, mixed number, and whole number combinations. Estimation is developed across the problems in the investigation, as well as the idea that multiplication does not always lead to a larger product. The last problem is structured like the one used to develop addition and subtraction algorithms. Students are given a set of presorted multiplication problems. Through the process of solving and characterizing how the problems are categorized, students develop a general algorithm for fraction multiplication.

Investigation 4

Dividing With Fractions

This investigation has the same goals as Investigations 2 and 3, except the operation of division is explored. Everyday situations are used to help students make sense of when division is an appropriate operation. The inverse relationship of multiplication and division is explored. The last problem uses presorted division problems to develop a general algorithm for fraction division.

Mathematics Background

Writing Number Sentences

Helping students learn to use mathematical language (i.e., symbols) correctly and with confidence is a goal of the CMP materials. We do this by using symbols connected to contexts so that the context gives the symbols meaning. Using symbolic notation, as well as reformulating symbolic expressions using the rules and syntax of mathematics, can give new insights into problem situations. These are among the fundamental activities of mathematics. Learning the symbolic language of mathematics requires experience to make sense of what the symbols mean and how to operate with them. In CMP, we develop symbolic proficiency over time. You will see students frequently asked to write a number sentence to capture their work. The sentences become more symbolic in grade seven and reach higher levels of sophistication in grade eight units.

Developing Algorithms

Rational numbers are the heart of the middle-grade experiences with number concepts. The concepts of fractions and operations on fractions can be difficult for students. Part of the reason for students' confusion about rational numbers can be a rush to symbol manipulation with fraction operations without time spent in making sense of the concepts and building experiences that show reasons for why the algorithms work. In addition, students need to understand the operations in ways that help them to judge what operation or combination of operations is needed in a given situation. This unit is designed to provide experiences in building algorithms for the four basic operations with fractions, as well as opportunities for students to consider when such operations are useful in solving problems. Building this kind of thinking and reasoning supports the development of skill with the algorithms. By the end of this unit, we expect students to have efficient algorithms for all four basic operations with fractions, including mixed numbers.

The development of algorithms in this unit draws upon concepts and procedures developed in *Bits and Pieces I*. In *Bits and Pieces I*, students developed an understanding of basic interpretations, models, equivalence, and ordering of rational numbers. In *Bits and Pieces II,* we draw upon this development and the models that were introduced—fraction strips or bars, number lines, grid-area, and partition models—because they connect directly to operations on rational numbers. See *Mathematics Background* in the Teacher's Guide of *Bits and Pieces I* for a full discussion of the concepts and models introduced.

For all four operations we use the same type of development. The development of algorithms for each operation and the understanding of those algorithms involve experiences with contextual problems, models, strategies, and estimation. Problems in context help students make sense of an operation and how the operation can be computed. The problem contexts lead students to model situations and to write number sentences that are representative of the particular situation so they begin to make sense of when an operation is appropriate. Students are often asked to make estimates and use them to decide if their models and symbolic work are reasonable. By analyzing the diagrams and models they develop and their resulting quantities, and relating this to their symbolic work and their estimates, students begin to develop algorithms for fraction operations. An underlying goal of all this work is learning to both write and read mathematical language.

For each operation, the last problem in the investigation asks students to analyze a set of sorted computation problems that seem to belong together. They compute the problems and then look for strategies that can be used in a case of that sort. These are refined into algorithms that are efficient for all cases. Usually the students end up with algorithms that are like the standard algorithms taught by direct instruction in many programs. But, they also end up with understanding and insight into the operations and when they are useful. Also, students will have useful strategies for computations with particular number situations. For example, students may come to realize that in a multiplication problem if one factor is $\frac{1}{2}$, you can compute the product by doubling the denominator of the other factor because the piece size needs to be half as big. $\frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$.

Estimation

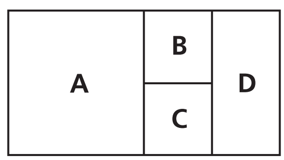
Rather than rush to compute in a given situation, students have experiences with estimating sums, differences, products, and quotients. The initial questions CMP helps students to ask are, "About how big will the answer be? What answer makes sense?" These give students a way to know if their computations are wrong, whether the calculation has been done by hand or by calculator. Developing such a sense of fractions and operations takes a long time. At this point in the curriculum students have had quite a bit of practice finding equivalent fractions and decimals and changing among fractions. They have developed some benchmark fractions that they can use to estimate relative size. Students will use these skills to develop strategies to estimate fraction computations.

The strategies used to estimate can differ. For example, in a situation where the goal is to decide what whole number a sum is closest to or what is a reasonable sum, a useful strategy is to round the numbers to the nearest benchmark. In contrast, you may want to use an estimation strategy that leads to an estimate that you know is too large (overestimate) or too small (underestimate). Consider the situation where you go shopping and you cannot spend more than \$20. When you estimate the total cost of the items you want to purchase, you need to estimate in such a way that you can be sure your actual sum is less than \$20. Using a purposeful estimation approach, in this case an overestimate, you can know whether or not your actual sum will be less than \$20.

Addition and Subtraction

Strategies for operations with fractions can be developed with contexts that help students learn how to put fractions together and take them apart. As students model and symbolize aspects of contextual situations, students develop meaning for and skill in using the operations of addition and subtraction. Additionally, students learn the value of equivalence when changing the representation of fractions to a form with common denominators so that the numerators can be added or subtracted.

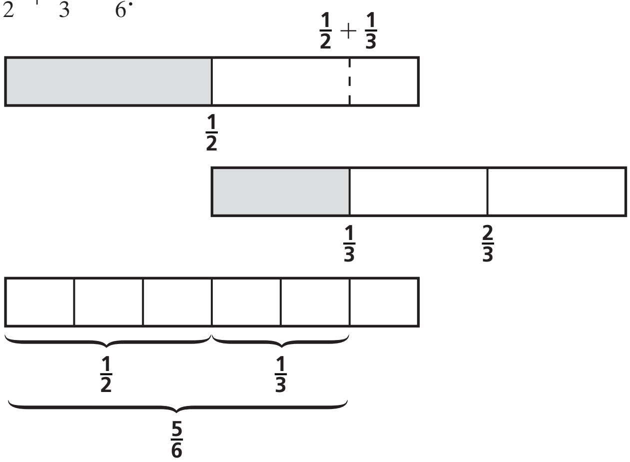
Students' previous work with equivalence and partitioning is critical to the development of strategies for adding or subtracting. The following area model provides a context where both naming fractional parts of a whole and equivalence can emerge as students try to write number sentences to model combining section A with section B.



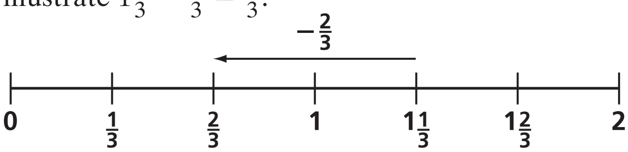
In order to find the sum of A + B, or $\frac{1}{2} + \frac{1}{8}$, students need to use equivalent fractions to rename $\frac{1}{2}$ as $\frac{4}{8}$. The area model helps students visualize A, $\frac{1}{2}$, as 4 eighth-sized sections. By asking students to write number sentences, and to explain how the number sentence helped them arrived at the sum $\frac{5}{8}$, students begin to understand why it is necessary to rename fractions when adding and subtracting and the role that equivalence plays in doing so.

In addition to equivalent fractions, students will need to draw on their understanding of equivalent forms. In *Bits and Pieces I* students worked with the relationship between a mixed number and an improper fraction. As students develop strategies to add and subtract fractions in situations that lead to borrowing or carrying, students learn the value of being able to rewrite fractions in equivalent forms. For example, understanding why $8\frac{2}{3}$ is equivalent to $7\frac{5}{3}$ is critical to understanding how to borrow in fraction situations.

There are other models that can be used to highlight the role of equivalence and support understanding of addition and subtraction. The *fraction-strip model* was used in conjunction with the *number-line model* in *Bits and Pieces I* to develop meaning for fractions and equivalence. Here fraction strips are used to represent $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.



The *number-line model* helps make the connection to fractions as numbers or quantities. This is a number line for 0 to 2 marked to illustrate $1\frac{1}{3} - \frac{2}{3} = \frac{2}{3}$.

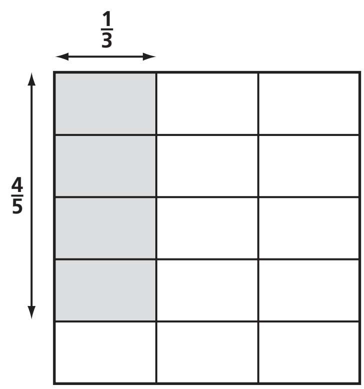


Multiplication

One of the first hurdles for students in their understanding of multiplication of fractions is realizing that multiplication does not always "make larger," as their experience with whole number multiplication has firmly established. In fact, with multiplication of a fraction by a whole number, the fraction can be interpreted as *an operator* that may "stretch" (make larger) or "shrink" (make smaller) depending on whether the fraction is greater or less than 1. This is a big idea that supports understanding what multiplication of fractions entails.

A second hurdle for students is understanding that when they encounter a situation where one needs to take a fraction *of* a quantity, *of* means multiplication. For example, to find $\frac{2}{3}$ of 9, you multiply $\frac{2}{3} \times 9$ to get 6. The temptation is very great to start by telling students this rather than have them encounter the dilemma of the meaning of *of*. In resolving what *of* means in this context, student learning is enhanced.

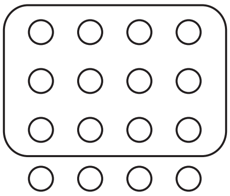
Models for multiplication of fractions used in the unit are both area models and partitioning. Area models are also useful to help students represent situations, especially multiplication and later decimals and percents. The diagram shows $\frac{4}{5}$ of $\frac{1}{3}$.



$\frac{4}{5}$ of $\frac{1}{3} = \frac{4}{15}$

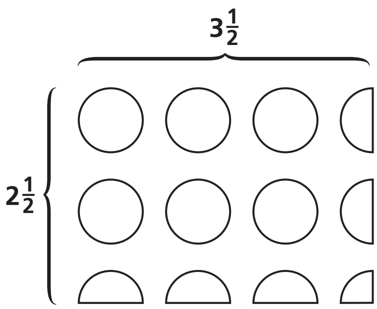
Students also use a model of fraction situations that is based on *partitioning* a number line or strip. The example shows finding $\frac{4}{5}$ of $\frac{1}{3}$ or $\frac{4}{5} \times \frac{1}{3}$. (Figure 1)

You may see students use discrete models to make sense of situations where they are working on discrete objects. An example of a discrete situation is finding $\frac{3}{4}$ of 16 apples. Here each apple represents a separate entity.



$\frac{3}{4}$ of 16 = 12

Another example, below, shows $2\frac{1}{2} \times 3\frac{1}{2}$.

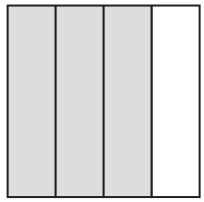


$2\frac{1}{2} \times 3\frac{1}{2} = 8\frac{3}{4}$

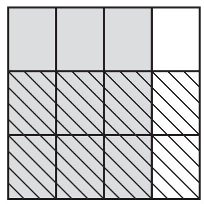
Developing the Multiplication Algorithm

Students notice that multiplication is easy for proper fractions because they can just multiply the numerators and multiply the denominators. However, they have little understanding of why this works. The models we have discussed can help you support understanding. The following looks at both the area and the number-line model as a means of understanding why the algorithm works.

Consider the problem $\frac{2}{3} \times \frac{3}{4}$. To show $\frac{2}{3} \times \frac{3}{4}$ with an area model, first represent the $\frac{3}{4}$ by dividing a square into fourths and shading three of the fourths.

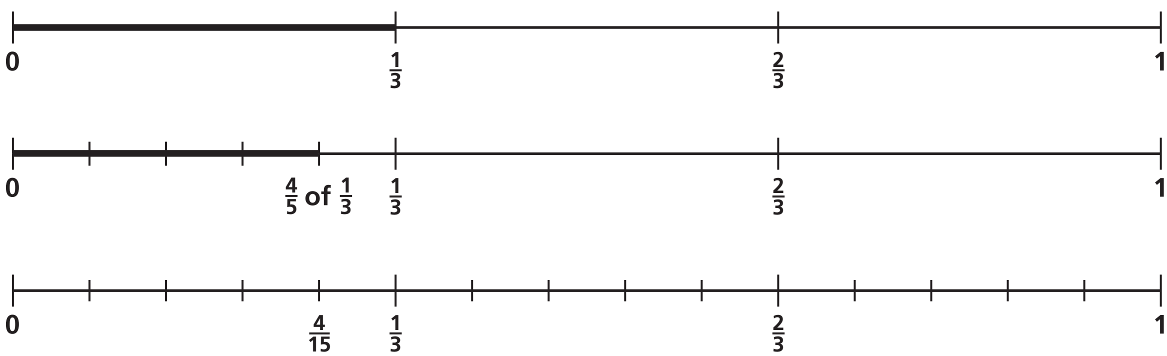


To represent taking $\frac{2}{3}$ of $\frac{3}{4}$, divide the whole into thirds by cutting the square the opposite way, then shade two of the three sections. The part where the shaded sections overlap represents the product, $\frac{6}{12}$.



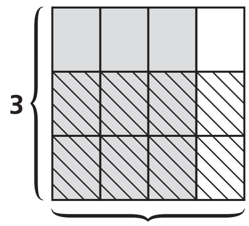
Note what happens to the numerator and the denominator when you partition and how this is related to the algorithm for multiplying fractions. When the square is partitioned, the denominators are used to partition and repartition the whole. In this problem, there are fourths or four parts. When the fourths are partitioned into thirds, or three parts each, the number of pieces in the

Figure 1



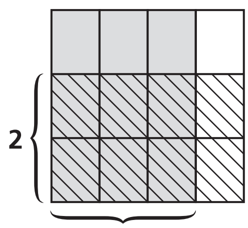
$\frac{4}{5}$ of $\frac{1}{3} = \frac{4}{15}$

whole triples so there are 12 pieces. In the algorithm, when you multiply the denominators (3×4), you are resizing the whole to have the correct number of parts. This means that the denominator in the product has the same role as the denominator in a single fraction. The role is to determine how many parts are in the whole.



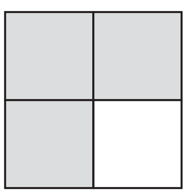
denominator → $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$

Likewise, the numerator is keeping track of how many of the one-twelfth parts are being referenced. During the original partitioning, $\frac{3}{4}$, or 3 fourth-sized parts, were shaded. In order to take $\frac{2}{3}$ of the 3 one-fourth sized parts, you have to take 2 of the one-twelfth sections from each of the 3 one-fourth sized parts. This can be represented by the product of the numerators 2×3.



numerator → $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$

Note that dividing a square with both horizontal and vertical lines for the first fraction does not lead to the kind of partitioning that suggests multiplication of numerators and denominators. For example, if you represent $\frac{3}{4}$ like this:

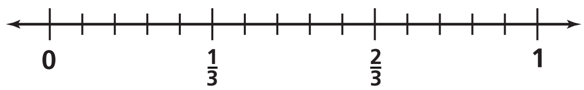


you may find $\frac{2}{3}$ of $\frac{3}{4}$ by noticing that there are three pieces shaded and you are concerned with 2 of them, so the answer is $\frac{2}{4}$. This is a perfectly reasonable strategy for this problem. The question is whether this strategy will *always* work no matter what fractions. For $\frac{1}{5} \times \frac{2}{3}$ this is not a helpful strategy.

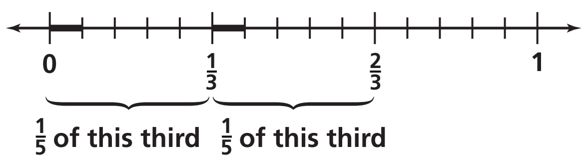
The following illustrates how the *number-line model* is helpful for $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$ and is generalizable, even if tedious with large numerators or denominators.

Draw a number line and label 0 and 1. Partition the number line into thirds and mark $\frac{1}{3}$ and $\frac{2}{3}$.

Now break each third into 5 equal parts to get a total of 15 equal parts.



Each fifth of a third is $\frac{1}{15}$, so the two parts marked would be $\frac{2}{15}$. Again the product of the numerators gives the numerator of the product and the product of the denominators gives the denominator of the product.



Using Distribution as a Strategy to Multiply Fractions

Another approach to multiplication of fractions that students use is based on the distributive property. This approach is often used when mixed numbers are involved. Many students use the ideas of breaking a number apart quite intuitively, but often do so incorrectly. The terminology of distribution is not important for students to know at this time. However, because students often use this strategy incorrectly, we provide an opportunity to talk about it in Problem 3.4. We do not wish to promote it as the only approach to multiplying fractions, but as one that is sensible in some situations.

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Here is an algorithm for whole number multiplication that uses this approach. Consider the problem 32×24.

$$\begin{array}{rl} 32 & \\ \times\ \underline{24} & \\ 8 & \leftarrow 4 \times 2 \\ 120 & \leftarrow 4 \times 30 \\ 40 & \leftarrow 20 \times 2 \\ \underline{600} & \leftarrow 20 \times 30 \\ 768 & \end{array}$$

This approach is very much like multiplying binomials in algebra. It involves breaking up both numbers into their respective tens and ones. With 32×24 it looks like this:

$$30 \times 20 = 600 \qquad (30 + 2) \times (20 + 4) \qquad 30 \times 4 = 120$$

$$2 \times 20 = 40 \qquad (30 + 2) \times (20 + 4) \qquad 2 \times 4 = 8$$

First, multiply 30 times 20 (tens place in 24) and then 30 times 4 (ones place in 24). Next, multiply by the 2 in the ones place of 32. Multiply 2 times 20 (tens place in 24) followed by 2 times 4 (ones place in 24). Finally, total each partial product $(600 + 120 + 40 + 8)$ to arrive at a total product of 768.

With a problem like $2\frac{1}{2} \times 2\frac{1}{4}$, students may break up each factor and try to work with $(2 + \frac{1}{2}) \times (2 + \frac{1}{4})$. If they distribute correctly, they would reason as shown next.

$$2 \times 2 = 4 \qquad (2 + \tfrac{1}{2}) \times (2 + \tfrac{1}{4}) \qquad 2 \times \tfrac{1}{4} = \tfrac{1}{2}$$

and

$$\tfrac{1}{2} \times 2 = 1 \qquad (2 + \tfrac{1}{2}) \times (2 + \tfrac{1}{4}) \qquad \tfrac{1}{2} \times \tfrac{1}{4} = \tfrac{1}{8}$$

$$(2 + \tfrac{1}{2}) \times (2 + \tfrac{1}{4}) \rightarrow 4 + \tfrac{1}{2} + 1 + \tfrac{1}{8} = 5\tfrac{5}{8}$$

Another approach that makes sense with this problem is to work with $(2 + \frac{1}{2}) \times 2\frac{1}{4}$. If you only break up the first factor the reasoning is as follows:

$$2 \times 2\tfrac{1}{4} = 4\tfrac{1}{2} \qquad (2 + \tfrac{1}{2}) \times 2\tfrac{1}{4} \qquad \tfrac{1}{2} \times 2\tfrac{1}{4} = 1\tfrac{1}{8}$$

$$(2 + \tfrac{1}{2}) \times 2\tfrac{1}{4} \rightarrow 4\tfrac{1}{2} + 1\tfrac{1}{8} = 5\tfrac{5}{8}$$

Division

Division also has its share of conceptual difficulties. The answer to a division involving fractions is not necessarily smaller than the dividend. Again it depends on the size of the fraction for both the dividend and the divisor. For example, $3 \div \frac{1}{3} = 9$ and $\frac{1}{4} \div \frac{1}{3} = \frac{3}{4}$ result in a quotient that is larger than the dividend or the divisor. However, in $\frac{1}{3} \div 9 = \frac{1}{27}$, the quotient is smaller than either the dividend or the divisor, and in $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$, the quotient is between the dividend and the divisor! Examination of division of fractions in context can help students build an understanding of the operation as well as skill in predicting (or estimating) the kind of answer expected.

In the development of meaning of operations, we ask students to write problems that fit a given computation expression. This will tell you a lot about whether students can interpret different kinds of division situations and whether they can make sense of what the answer to a division problem, including its fractional part, means in a given situation.

In order for students to make sense of any division algorithm, they need to think about what the problem is asking. Creating diagrams to model division problems is a key part of developing this understanding. There are cases where the use of pictorial reasoning is more efficient or just as efficient as an algorithm. Also, the development of an efficient algorithm is tied to one's ability to understand pictorially and linguistically what the problem is asking. As students work toward trying to develop and use algorithms they may continue to draw pictures to help them think through the problem. However, they also need to learn to talk about what the problem is asking, what the answer means, what makes sense as a solution strategy, and how this language is related to the algorithm.

Our goal is to help students develop an efficient algorithm. Not all students may get to the "reciprocal" algorithm for division of fractions, but they should have efficient strategies that make sense to them to solve problems that call for division with fractions.

Understanding Division as an Operation

There are two situations associated with division. We can focus on division as a *sharing* operation in problems like this:

> Ms. Li brings peanuts to be shared equally by members of groups winning each game. How much of a pound of peanuts will each student get when the peanuts weigh $\frac{1}{2}$ pound and four students are on the winning team?

Here the question is how much each of the four team members will get if the amount is shared equally. You can also think of this as a *partitioning,* sometimes called *partitive,* model.

Another kind of situation calling for division is a *grouping* situation. For example:

> Naylah plans to make small cheese pizzas to sell at a school fundraiser. She has nine bars of cheese. How many pizzas can she make if each takes $\frac{1}{3}$ bar of cheese?

Here the question is how many groups of size $\frac{1}{3}$ can be made from nine bars of cheese? Another way to ask this is "How many $\frac{1}{3}$'s are in 9?" This kind of problem has multiple names—*measurement, subtractive,* or *quotitive* model. Knowing these names is not important for your students, but it is important for them to experience situations representing these different interpretations of division. Otherwise students will not have all the tools for deciding *when* division is the appropriate operation.

Developing a Division Algorithm

We develop understanding of division of fractions by looking at three cases—division of a whole number by a fraction, division of a fraction by a whole number, and division of a fraction by a fraction. From these situations, several approaches to division are developed: multiplying by the denominator and dividing by the numerator, multiplying by the reciprocal, and the common denominator approach.

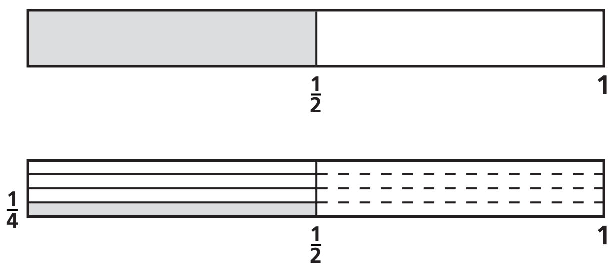
Multiplying by the Denominator and Dividing by the Numerator When you confront a whole number divided by a fraction, such as $9 \div \frac{1}{3}$, it is easiest to interpret this as finding how many $\frac{1}{3}$'s are in 9. To answer, students find how many $\frac{1}{3}$'s are in a whole and multiply by 9 to find the total number of $\frac{1}{3}$'s in 9. The reasoning is as follows: In $9 \div \frac{1}{3}$, I have to find the total number of $\frac{1}{3}$'s in 9. I know that there are three $\frac{1}{3}$'s in 1, so there are 9×3 or 27 of the $\frac{1}{3}$'s in 9. In summary, $9 \div \frac{1}{3} = 9 \times 3 = 27$.

Next we move to $9 \div \frac{2}{3}$. A key to understanding in the development of division of fractions is the relationship between the two problems $9 \div \frac{1}{3}$ and $9 \div \frac{2}{3}$. The question is, how are the answers related and why? The answer to the first problem is 27 and the answer to the second is $13\frac{1}{2}$. Why does it make sense for the answer to the second to be half that of the first? You can interpret the first problem as how many $\frac{1}{3}$'s are in 9 and the second as how many $\frac{2}{3}$'s are in 9. Now it makes sense that it will take twice as much to make $\frac{2}{3}$ than to make $\frac{1}{3}$ so the number you can make will be half as large.

This conversation allows students to begin to relate a whole string of division problems, such as $9 \div \frac{1}{3}$, $9 \div \frac{2}{3}$, $9 \div \frac{3}{3}$, and $9 \div \frac{4}{3}$. Here we are building a case for thinking of division of fractions as multiplying by the denominator of the divisor to find how many in one whole and then dividing by the numerator because that is how many it takes to make a piece of the required size. These two actions are the same as multiplying by the reciprocal.

When moving to other cases, such as dividing a fraction by a whole number and dividing a fraction by a fraction, support student thinking with models. In these situations, we continue to see that multiplying by the denominator and dividing by the numerator makes sense because we can interpret what each part is accomplishing. In the computation $\frac{2}{3} \div \frac{3}{4}$, we can find the "answer" by multiplying by $\frac{4}{3}$. But what does this mean? Multiplying by 4 tells us how many $\frac{1}{4}$'s are in a whole and dividing by 3 adjusts this answer by accounting for the fact that it takes 3 of the $\frac{1}{4}$'s to make one object of the size the problem requires. We have found that many students are able to see the pattern of "multiply by the denominator and divide by the numerator of the divisor" and explain why it makes sense through this kind of classroom talk.

Multiplying by the Reciprocal The reciprocal approach may arise when working with fraction divided by whole number contexts. For example, with the problem $\frac{1}{2} \div 4$, students often draw the following diagram.



They may reason by saying, "I divided the $\frac{1}{2}$ into four parts so I could find $\frac{1}{4}$ of the half." Here students are relating the problem $\frac{1}{2} \div 4$ to the problem $\frac{1}{2} \times \frac{1}{4}$. This type of reasoning, the diagram that develops it, and the number sentences that support it, help students move from the division problem to multiplying by the reciprocal.

Common Denominator Approach Some students intuitively try the same approach for division that worked in addition and subtraction—finding a common denominator. This algorithm nicely links to their whole number understanding of division. For example, in the problem $\frac{7}{9} \div \frac{1}{3}$, students rename the fractions to say $\frac{7}{9} \div \frac{3}{9}$. The common denominator allows them to reason that if they have 7 one-ninth sized pieces of something and want to find out how many groups of 3 one-ninth sized pieces they can make, they can find the answer from the computation $7 \div 3$, which equals $2\frac{1}{3}$. This algorithm is used in *Bits and Pieces III* to develop decimal division.

Relating Multiplication and Division

In additive situations, those involving addition and subtraction, the quantities are easy to count, measure, combine and separate. This is because each quantity in an addition or subtraction problem has the same kind of label or is the same type of unit. For example, $3 + 4 = 7$ can be thought of as 3 marbles plus 4 marbles equals 7 marbles. Each quantity is a number of marbles.

In multiplicative situations, those involving multiplication and division, the quantities are not so straightforward. Each number may represent a different kind of unit. For example, if tomatoes cost $0.87 per can, the total cost for 6 cans can be found by multiplying 6 cans $\times$ 87 cents. It is hard to imagine a situation where adding tomatoes and money would make sense.

Another challenge is the different kinds of situations that call for multiplication and for division. A multiplication problem may be counting an array, or finding an area, or finding the sum of a repeated addition, and so on. Division may be finding how many groups of a certain size or measure that you can make from a given quantity or how many objects or parts would be in each of a given number of groups.

For example, the number sentence $3 \times 4 = 12$ could represent 3 people, each with 4 candies. The same number sentence could also represent 3 candies given to each of 4 people. The two types of division situations, sharing and grouping, are related to these two multiplication situations. The diagram in Figure 2 shows the grouping model of division first followed by the sharing model.

It is important that students develop a sense of the kinds of situations for which each operation is useful. Therefore, you will see attention to meanings and interpretation of the operation in the unit.

Inverse Relationships

Fact families and missing-value problems are used to introduce the inverse relationships of addition and subtraction, and multiplication and division. In elementary grades, students learn about *fact families* for whole numbers. For example, they learn that the addition problems $3 + 5$ and $5 + 3$ both equal 8. In addition, they are related to two subtraction sentences, $8 - 5 = 3$ and $8 - 3 = 5$. In this unit these ideas are expanded to include fractions. For example, students learn that the addition sentence $\frac{7}{10} + \frac{1}{2} = \frac{6}{5}$ is related to $\frac{1}{2} + \frac{7}{10} = \frac{6}{5}$ and two subtraction sentences, $\frac{6}{5} - \frac{1}{2} = \frac{7}{10}$ and $\frac{6}{5} - \frac{7}{10} = \frac{1}{2}$.

Understanding the inverse relationship between the operations pairs of addition-subtraction and multiplication-division is a tool that lends itself to many situations, one of which is equation solving. In this unit, missing-value problems are used to introduce students to the use of a variable as a placeholder. However, the focus is on understanding inverse relationships. We do not expect students to develop formal procedures or notation for solving algebraic equations at this stage.

Missing-value problems, as used in this unit, will help students begin to develop a generalized understanding of inverse relationships. This generalization is aided by working on these relationships in non-whole number contexts. In whole number contexts, such as $20 \div N = 5$, solving for N is partly aided by students using multiplication and division facts with which they had repeated experience. In a problem like $\frac{8}{15} \div N = \frac{2}{3}$, this becomes more difficult. Students have to think about which values are the factors in the related multiplication problems $\frac{2}{3} \times N = \frac{8}{15}$ and $N \times \frac{2}{3} = \frac{8}{15}$. Here, N and $\frac{2}{3}$ are the factors and $\frac{8}{15}$ is the product of the related multiplication problem. Going a step further, if you divide the term that represents the product of the multiplication problem by one of the factors you will get the other factor. This leads to the related missing-value problem $\frac{8}{15} \div \frac{2}{3} = N$ or the realization that if you divide $\frac{8}{15}$ by the known value $\frac{2}{3}$, you will get the other factor, N.

Keep in mind that inverse relationships will be explored in later units with other number contexts such as decimals and integers. For most students this will be an initial introduction and mastery is not expected. But over time, students will start to think beyond the actual numbers to the relationships that exist among the values in related addition and subtraction problems and multiplication and division problems, or fact families. This understanding will be a powerful tool for students to use in other mathematical contexts.

Figure 2

Grouping

3	×	4	=	12	→	12	÷	4	=	3
Unit		*Rate*		*Total*		*Total*		*Rate*		*Unit*
Number of people		Candy per person		Candies		Candies		Candy per person		Number of people

Sharing

3	×	4	=	12	→	12	÷	4	=	3
Rate		*Unit*		*Total*		*Total*		*Rate*		*Unit*
Candy per person		Number of people		Candies		Candies		Number of people		Candy per person

Content Connections to Other Units

Big Idea	Prior Work	Future Work
Performing mathematical operations with fractions	Interpreting fractions as part-whole relationships; combining and comparing fractions, partitioning and repartitioning fractions, finding equivalent fractions (*Bits and Pieces I)*; factorization of numerators or denominators (*Prime Time)*	Interpreting fractions as probabilities (*How Likely Is It?)*; interpreting fractions as scale factors, ratios, and proportions (*Stretching and Shrinking*); interpreting fractions as constants and variables in linear and nonlinear equations and relationships (*Variables and Patterns; Moving Straight Ahead; Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols*); using fractions to help understand irrational numbers (*Looking for Pythagoras*); using fractions to understand integers (*Accentuate the Negative*); interpreting and applying fractions (*Bits and Pieces III; What Do You Expect?; Samples and Populations*)
Developing and applying algorithms for performing fraction calculations	Estimating to check reasonableness of answers (*Bits and Pieces I)*	Developing algorithms for finding the area and perimeter of 2-D shapes (*Covering and Surrounding*); developing algorithms for finding the volume and surface area of 3-D shapes (*Filling and Wrapping*); developing algorithms for integer computation (*Accentuate the Negative*); developing algorithms for decimal computation (*Bits and Pieces III*); applying fractions in studying probability (*How Likely Is It?; What Do You Expect?; Samples and Populations*); applying ratios, proportions, and scale factors (*Stretching and Shrinking; Comparing and Scaling*)
Inverse relationships and operations in fraction settings	Inverse operations in whole number settings (elementary school)	Inverse operations in decimal settings (*Bits and Pieces III*); inverse operations in integer settings (*Accentuate the Negative*); finding an unknown dimension given area or volume (*Covering and Surrounding; Filling and Wrapping*); solving algebraic equations (*Moving Straight Ahead; Thinking With Mathematical Models; Say it With Symbols*); patterns of change (*Moving Straight Ahead; Thinking With Mathematical Models*)
Performing mathematical operations with fractions	Interpreting fractions as part-whole relationships; combining and comparing fractions, partitioning and repartitioning fractions, finding equivalent fractions (*Bits and Pieces I*)	Interpreting fractions as probabilities (*How Likely Is It?*); interpreting fractions as scale factors, ratios, and proportions (*Stretching and Shrinking*); interpreting fractions as constants and variable coefficients in linear and nonlinear equations and relationships (*Variables and Patterns; Moving Straight Ahead; Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols*); using fractions to help understand irrational numbers (*Looking for Pythagoras*); using fractions to understand integers (*Accentuate the Negative*); interpreting and applying fractions (*Bits and Pieces III; What Do You Expect?; Samples and Populations*)

Overview

Rational numbers and their various forms of representation and interpretation are the heart of the middle-grades experiences with number concepts. In earlier units, students have explored various meanings of and models for rational numbers in fraction form and in decimal form. They have developed efficient algorithms for addition, subtraction, multiplication, and division with fractions. These algorithms will be one basis on which understanding of operations with decimals will be built. Here the meaning of decimals as special fractions with denominators that are powers of 10 will be the focus and help make the connection.

In elementary school, the place value chart is often extended to include digits to the right of the decimal point. However, this knowledge is very fragile for most students in grade six. Using a place value interpretation alone places a great deal of faith in number patterns to help students make sense of operations, especially of multiplication and division. We have chosen instead to use both a fraction interpretation and a place value interpretation of decimals to support the development of algorithms.

This unit is designed to provide experiences in building algorithms for the four basic operations with decimals, as well as opportunities for students to consider when such operations are useful in solving problems. For example, what signals indicate to the student that division will help solve a problem? Building this kind of thinking and reasoning supports the development of skill with the algorithms. We build on students' familiarity with money as an entry point and use other familiar measurement situations where measures are given in decimals.

As the title also implies, this unit uses the students' knowledge of operations with decimals to return to percents and to further develop students' understanding and skill in solving percent problems. Particular attention is paid to solving the relationship *a% of b equals c*, when only two of the three values *a, b,* and *c* are given. In many texts, the three cases for the missing variable are taught as separate, unrelated problems. We want students to see that finding a percent of a number, finding what percent one number is of another number, and finding the original number if you know a percent of the number are all versions of this basic relationship with different unknown variables. In the case of finding a percent of a number, the unknown is *c,* for example, 5% of 24 equals *c*. In finding what percent one number is of another, the unknown is *a*, for example, *a*% of 48 equals 6. And, in finding the original number if you know the percent of the number, the unknown is *b,* for example, 15% of *b* equals 27.

Before we look at the specific mathematical ideas developed and used in this unit, it is helpful to review the ideas developed in *Bits and Pieces I* and *II* as these are the underpinnings for this unit.

CMP and Fractions: Review of *Bits and Pieces I* and *II*

In CMP, we have developed a set of connected units that comprise the rational number strand for grade six. In *Bits and Pieces I,* the first unit on fractions, decimals, and percents, the investigations ask students to make sense of the meaning of fractions, decimals, and percents in different contexts. The unit emphasizes developing an understanding of basic interpretations, models, equivalence, and ordering of rational numbers. Students learn to move among equivalent forms of fractions and to move among fractions, decimals, and percents. They also build benchmarks for estimating locations of rational numbers on a number line and begin to estimate simple sums and differences. The models introduced and used in *Bits and Pieces I* (fraction strips, bars, number lines, and area models) are continued and built upon in *Bits and Pieces II.*

In *Bits and Pieces II,* students develop algorithms for fraction computations. As is the case with other aspects of CMP, students are confronted with situations that call for putting together, taking apart, duplicating, counting an array, sharing, grouping, partitioning, measuring, etc. As they confront such situations, they not only learn to "do" addition, subtraction, multiplication, and division of fractions, but they also learn the meaning of the operations and the kinds of situations that call for each. As students work individually, in groups, and as a whole class on problems, they learn ways of thinking about and operating with fractions and they practice the algorithms to develop skill in carrying them out.

We expect students to finish *Bits and Pieces II* knowing algorithms for computation that they understand and can use with facility.

Interpretations of Fractions

The major interpretations of fractions students encounter in *Bits and Pieces I* and *II* are:

1. fractions as parts of a whole
2. fractions as measures or quantities
3. fractions as indicated division

Interpretations such as fractions as operators ("stretchers" or "shrinkers") and fractions as rates, ratios, or parts of a proportion are foreshadowed here and continued in later grades. For a fuller discussion of these ideas, please see the Teacher Guides to *Bits and Pieces I* and *II*.

Models of Fractions

The models of rational numbers used throughout CMP were chosen because they connect directly to important interpretations of rational numbers. The fraction models used for developing both meaning and the operations on them are:

1. fraction-strip models
2. number-line models
3. grid-area models
4. partition models

For a fuller discussion of these models please see the Teacher Guides to *Bits and Pieces I* and *II*.

Summary of Investigations

Investigation 1

Decimals—More or Less!

Investigation 1 develops addition and subtraction of decimals. One problem focuses on estimation strategies, as do other problems in the unit. The initial questions CMP helps students to ask are, "About how great will the answer be? What makes sense?" These give students a way to know if their computations, done by hand or by calculator, are at least close to the correct answer or obviously wrong. In many situations an estimate is sufficient to "solve" the problem or make the needed decision.

Other problems in Investigation 1 focus on the place value interpretation of a number and what that means for adding or subtracting numbers.

Addition-subtraction fact families are used to help solve for missing addends or sums in situations written in symbolic form. Students write mathematical sentences using symbols to indicate the required computation(s). An underlying goal of all this work is learning both to write and to read mathematical language. Additionally, students learn the value of changing the representation of fractions and decimals that they need to add or subtract to a form with common denominators, so that the numerators can be added or subtracted. Students' previous work in locating and representing fractions on the number line is critical to the development of common denominators as a strategy for adding or subtracting. In the end, students articulate an algorithm for adding and subtracting.

Investigation 2

Decimal Times

This investigation focuses on developing an algorithm for multiplying decimals. Students use fractions to help make sense of multiplication of decimals. They look at products, find missing factors, and use estimation as a way to determine where the decimal has to be in a product of decimal numbers. Problem 2.4 lays the groundwork for the simple shortcut algorithm: Multiply the decimals as whole numbers and adjust the place of the decimal in the product.

Investigation 3

The Decimal Divide

Investigation 3 develops an algorithm for division of decimals. In developing the algorithm, students solve a set of contextualized problems that provide a common sense way to think about decimal division based on what they already know about whole-number and fraction division. Students use the fraction form of decimals to develop an algorithm for dividing decimals. The last two problems look at patterns in division and in terminating and repeating decimals.

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Investigation 4

Using Percents

In all three problems of Investigation 4, students look at real situations in which one encounters percents. The typical situations of discounts, taxes, and tips help students think about taking a percent of a number. The discount and tax situations help students to consider the amount left when a reduction is made and the total when taxes are added.

Investigation 5

More About Percents

In this investigation, students are asked to devise a general strategy for finding a percent when they are dealing with totals that are more than or less than 100.

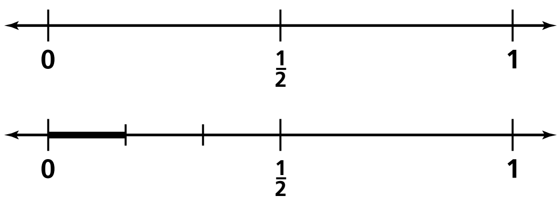
Mathematics Background

The following are key ideas in developing algorithms for fractions that are continued in *Bits and Pieces III.*

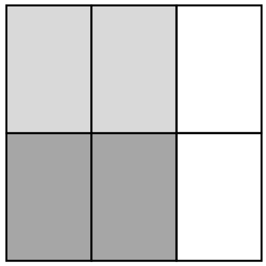
Decimal Multiplication and Division

One of the first hurdles for students in their understanding of multiplication of fractions and decimals is realizing that multiplication does not always "make greater." Their experience with whole-number multiplication has firmly established this incorrect belief. In fact, multiplication involving numbers that are not whole numbers may be interpreted as an operator that may "stretch" (make greater) or "shrink" (make less) depending on whether the fraction or decimal is greater or less than one. This is a big idea that supports understanding what multiplication of fractions or decimals entails.

Models for multiplication of fractions used in the *Bits and Pieces II* unit are both partitioning and area models. An example of partitioning is: Seth is running $\frac{1}{3}$ of a $\frac{1}{2}$-mile relay race. How far will he run?



An example of an area model is: Mr. Sims asks to buy $\frac{1}{2}$ of a pan that is $\frac{2}{3}$ full. What fraction of a whole pan does Mr. Sims buy?



These models continue to be useful with decimals.

Division also has its share of conceptual difficulties. The answer to a division problem involving fractions, whether in fraction or decimal form, is not necessarily less than the dividend. Again, it depends on the size of the divisor. For example, the answer to $3 \div 0.25 = 12$ is greater than the dividend. Another example is $0.25 \div 0.4 = 0.625$. What matters is that the divisor is less than 1.

Decimal Estimation

In *Bits and Pieces I* we looked at benchmark fractions and their decimal equivalents. These fraction benchmark ideas can be used to estimate small decimal computations as well. For example, let us look at estimating the sum, the difference, the product, and the quotient of 0.78 and 0.14.

$0.78 + 0.14$ is near $\frac{3}{4} + \frac{1}{8}$ or $\frac{6}{8} + \frac{1}{8}$. So a reasonable estimate is a little less than 1 or perhaps about 0.9.

$0.78 - 0.14$ would be about $\frac{6}{8} - \frac{1}{8}$ or $\frac{5}{8}$. So a reasonable estimate is about 0.6.

0.78×0.14 would be about $\frac{6}{8} \times \frac{1}{8}$ or $\frac{6}{64}$. So a reasonable estimate is about 0.1.

$0.78 \div 0.14$ would be about $\frac{6}{8} \div \frac{1}{8}$ or about 6.

In each case estimating with benchmark fractions is helpful.

Other strategies are also useful. In the examples above, students can round to convenient decimals in their heads and estimate from these. For example, you could round 0.78 to 0.8 and 0.14 to 0.1. This rounds one number up and the other down. This would give the following mental computations with one-digit decimals to make an estimate:

$0.78 + 0.14$ is near $0.8 + 0.1$ but a bit greater than 0.9.

$0.78 - 0.14$ would be near $0.8 - 0.1$ but a bit less than 0.7.

0.78×0.14 would be about 0.8×0.1 or a bit greater than 0.08. So you might guess that the product is closer to 0.1.

$0.78 \div 0.14$ would be about $0.8 \div 0.1$ or about 8.

Since the dividend was rounded up and the divisor was rounded down, the estimate will be too large. So we might estimate the answer to be closer to 6.

There is no "one right way to estimate" that works for every situation. Students need to build a repertoire of strategies and a sense, through experience, discussion, and analysis, of what works in a given situation.

Developing Algorithms for Computing With Decimals

As we stated earlier, students have two ways of making sense of what decimals mean. Students extend the place value system on which our number system is built, or they can interpret decimals as fractions. (Obviously these two ideas are related, but they have different looks and feels to students.) In order to have the most robust understanding of and skill with computation, students need to understand each of these meanings of decimals and be able to use them. Depending on the operation, the fraction interpretation or the place value interpretation may contribute more directly to finding shortcut algorithms. However, looking at the algorithms developed through each lens can help develop deeper understanding.

The location of a digit in a number shows the value of the digit. This is a fundamentally important concept for students. Without place value understanding, work with decimals will suffer. In the Getting Ready for Problem 1.2, students look at why it makes sense to add by examining the value of each digit and recognizing that you must be careful to add digits that represent comparable values.

Looking at the patterns in such problems as the following, (Problem 2.4) brings a *place value* perspective to developing an algorithm for multiplication of decimals.

1. Find the following products using the fact that $21 \times 11 = 231$.

a. 2.1×11 **b.** 2.1×1.1

c. 2.1×0.11 **d.** 2.1×0.011

e. 0.21×11 **f.** 0.021×1.1

g. 0.021×0.11 **h.** 0.21×0.011

2. Test the algorithm you wrote in Question C on these problems.

We continue to use a combination of place value and fraction interpretations of decimals to develop a division algorithm. We also help students relate division of decimals to the long form of division of whole numbers.

Decimal Forms of Rational Numbers

Students have already observed that the decimal forms for some fractions, such as $\frac{1}{3} = 0.3333333\ldots$, "go on forever," but show a repeating pattern. Non-repeating infinite decimals such as 0.1011011101111011111 0... *never* reach a point where the digits start to repeat. These are *irrational numbers*, such as π and $\sqrt{2}$, and they are not considered in this unit. They are discussed in the grade 8 unit, *Looking for Pythagoras*. Here we are interested in the decimal forms of *rational* numbers.

Some rational numbers have a finite (or terminating) decimal form. Here are some examples: $\frac{1}{2} = 0.5$, $\frac{3}{4} = 0.75$, $\frac{1}{8} = 0.125$, $\frac{3}{25} = 0.12$. Others have an infinite repeating decimal form, such as $\frac{2}{3} = 0.666666666\ldots$, $\frac{8}{15} = 0.5333333$, or $\frac{3}{7} = 0.42857142857142\ldots$.

In Problem 3.5, we examine rational numbers to figure out how to predict whether a given fraction will have a repeating or terminating decimal form. Rational numbers in *simplified fraction form* that have only 2's or 5's in the prime factorization of the denominator will have a terminating decimal form, for example, $\frac{12}{75} = 0.16$. In simplified fraction form, $\frac{12}{75} = \frac{4}{25}$, which has only factors of five in the denominator. Fractions with factors other than 2's and 5's in the simplest denominator equivalent form will have a repeating decimal form, for example $\frac{13}{75} = 0.17333333\ldots$ and $\frac{4}{3} = 1.333333\ldots$.

Percents

This unit uses the students' knowledge of operations with decimals to return to the uses of percents and further develops students' understanding and skill in solving percent problems. Using the relationship *a*% of *b* equals *c*, any one of the letters *a*, *b*, or *c* can be the missing value. This means that the three kinds of percent problems developed separately in some texts are looked at in context with the focus on the relationships among these three variables. The following are examples of problems students solve in the fourth investigation of this unit.

1. Jill wants to buy a CD that is priced at $7.50. The sales tax is 6%. What will be the total cost of the CD?

In this problem, students know the price of the item and need to find 6% of the price. So here the values of *a* and *b* are known and the students must find the value of *c*. The equation would look like the following:

$$6\% \text{ of } \$7.50 = c.$$

They have to multiply 0.06×7.50 to find *c*.

2. Customers leave Jerome $2.50 as a tip for service. The tip is 20% of the total bill for their food. How much is the bill?

In the equation we now have 20% of *b* equals $2.50 and we have to find the value of *b*. Solving this equation can be done in several ways. One way is to ask how many 20%'s it takes to make 100%. In this case we need five. So $5 \times \$2.50$ gives us $12.50. Later in CMP, students will have more sophisticated equation-solving techniques and will be able to think of solving a problem like this by dividing each side by 0.2 to get $b = \$12.50$.

3. At another music store, Rita gets a $12 discount off a purchase of $48. What percent discount does she get?

In this situation our equation looks like this: *a*% of $48 equals $12. We have to find the percent that 12 is of 48. Students can informally solve this by asking themselves how many 12's it takes to get 48. It takes four 12's to make 48 so the percent must be $\frac{1}{4}$ of 100%. This would be 25%. As with the previous example, students will have more sophisticated solution methods later in the CMP curriculum and can return to these types of equations and divide each side of the equation by 48 to find 0.25, or 25%, as the answer.

These informal equation-solving techniques are powerful ways of thinking that are based on understanding the situation. Work of this kind should lead to better student monitoring of their work when equation-solving techniques are more fully developed in grades 7 and 8. Rushing to techniques here may mask understanding of the problem situations and what the problem is asking.

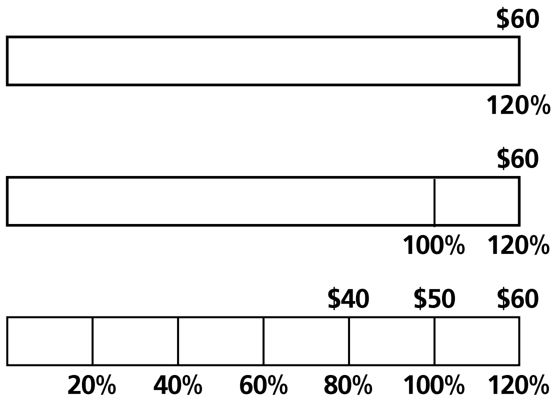
Working Backwards

A final kind of percent situation to which students are introduced is set in a restaurant scenario in which you know the total amount of money you have. You know what percent the taxes and the tip are, but you want to figure out how much you can spend on food and still pay the bill. Here is a specific example: Your group has $60. The tax on food is 5% and you want to leave a 15% tip on the food before tax. How much can you spend on food?

The first thing to realize is that you can add together the 5% and the 15% since you are taking these percents of the same number and adding them together ($0.15x + 0.05x = 0.20x$). If the tip were also on the tax, the situation would be more complicated.

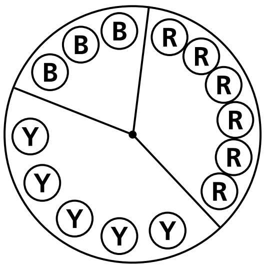
Here is a diagram that illustrates a way of solving the problem. Since the tax and tip will be 20% of the cost of the meal, the total amount can be thought of as representing 120% and the meal cost representing 100%. This leads to partitioning the 120% bar into six equal parts, each of which is 20%. On the cost line, the six partitions show us that each 20% represents $10, so the cost of the food must not exceed $50.

Series of percent bars



Circle Graphs

The final problem in the unit uses what students know about percents to make a new form of graphical representation for data called a circle graph. The key to a circle graph is that you know there are 360° in a full turn around the center of a circle. To represent the data, you need to figure out what angle represents the amount of turn for a certain percent of the data. One way to help students understand circle graphs is to start with small disks or other small objects, such as peas that have been colored. The total for the colors should be in different proportions. Form a circle with the objects making sure to put all the objects of one color adjacent to each other. Then draw the outline of the circle around the objects and draw lines from the center of the circle to separate the segments made by each color. This makes a rough circle graph.



There are 14 disks in the circle. Count the number of each color. Use fractions to represent the fractional part of the data each color represents.

$\frac{3}{14}$ blue; $\frac{5}{14}$ yellow; $\frac{6}{14}$ red

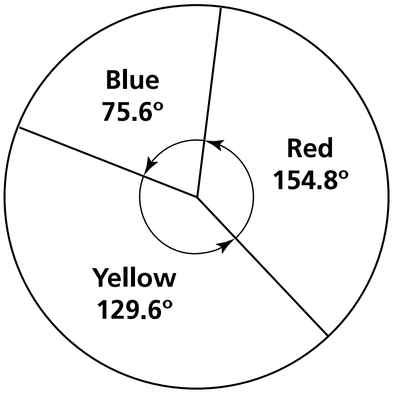
Use the equivalent percents to multiply by 360°.

$\frac{3}{14} \approx 21\%$ and $0.21 \times 360° = 75.6°$

$\frac{5}{14} \approx 36\%$ and $0.36 \times 360° = 129.6°$

$\frac{6}{14} \approx 43\%$ and $0.43 \times 360° = 154.8°$

Now make the circle graph precisely.



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Content Connections to Other Units

Big Idea	Prior Work	Future Work
Performing mathematical operations with decimals	Interpreting decimals as fractions; understanding place value of decimals; combining and comparing decimals (*Bits and Pieces I*); performing mathematical operations with fractions (*Bits and Pieces II*)	Interpreting decimals as probabilities (*How Likely Is It?; What Do You Expect?; Data Distributions; Samples and Populations*); interpreting decimals as scale factors, ratios, and proportions (*Stretching and Shrinking*); using decimals in scientific notation (*Data Around Us ©2004*); interpreting decimals as constants and variable coefficients in linear and nonlinear equations and relationships (*Variables and Patterns; Moving Straight Ahead; Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols; The Shapes of Algebra*); using decimals to understand negative rational numbers (*Accentuate the Negative*)
Developing and applying algorithms for performing decimal calculations	Connecting fractions, decimals, and percents to check the reasonableness of answers, estimating to check reasonableness of answers (*Bits and Pieces I*); developing and applying algorithms for performing fraction calculations (*Bits and Pieces II*); developing algorithms for finding the area and perimeter of 2–D shapes (*Covering and Surrounding*)	Developing algorithms for finding the volume and surface area of 3–D shapes (*Filling and Wrapping*); developing algorithms for integer computation (*Accentuate the Negative*); Applying decimals in studying probability (*How Likely Is It?; What Do You Expect?; Samples and Populations*); applying ratios, proportions, and scale factors (*Stretching and Shrinking; Comparing and Scaling*)
Inverse relationships and operations in decimal settings	Inverse operations in whole-number settings (elementary school); inverse operations in fraction settings (*Bits and Pieces II*); finding an unknown dimension given area (*Covering and Surrounding)*	Inverse operations in integer settings (*Accentuate the Negative; Filling and Wrapping*); solving algebraic equations (*Moving Straight Ahead; Thinking With Mathematical Models; Say It With Symbols; The Shapes of Algebra*); patterns of change (*Moving Straight Ahead; Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; The Shapes of Algebra*)
Performing computations involving percents	Defining, comparing, and applying percents (*Bits and Pieces I*)	Interpreting percents as probabilities (*How Likely Is It?; What Do You Expect?; Samples and Populations*); applying percents to analyze data (*Data About Us; Data Distributions*)

Overview

Although quantitative problems can be solved simply by counting members of a set or by measuring, it is often necessary to make decisions that involve comparisons of counts or measurements. The basic step in this kind of thinking is developed in elementary grades when such comparisons are decided by finding which number is greater. However, more useful reasoning often requires more careful comparison—explaining how much greater one number is than another, not in an absolute sense, but in a relative sense. There are many standard ways to make such comparisons (for example, fractions, ratios, rates, differences, and percents). One of the fundamental goals of school mathematics, especially middle-grades mathematics, is to help students develop flexible understanding, skill, and disposition in using strategies for comparing quantities. This goal runs throughout the Problems, ACE, and Reflections of this unit. The unit confronts students with a series of mathematical tasks that encourage them to make decisions about the quantities relevant to each task, how those quantities can be compared most usefully, and what information is provided by various quantitative comparisons.

The second major theme of this unit, as the title suggests, is *scaling*. In its most familiar sense, scaling suggests making something bigger or smaller, but similar in key respects to an original. Ratios and fractions often express comparative information in scaled-down form. For example, if a class consists of 15 boys and 10 girls, we might say that the ratio of boys to girls is 3 to 2, or that $\frac{3}{5}$ of the class is boys. We could also say that 60% of the class is boys, a kind of scaling up. *Stretching and Shrinking* lays a solid foundation of visual imagery to support the basic notion of scaling.

Research on students' understanding of proportional reasoning shows that moving from additive reasoning to multiplicative reasoning is difficult for students. Having experiences with geometric instances of proportional reasoning before concentrating on more numerical situations helps students in two ways: it gives students concrete experiences with visual representation of ratio comparisons, and it begins the work of helping students see the difference between reasoning by taking differences and reasoning by comparing ratios. This is why *Stretching and Shrinking* is in the CMP curriculum before *Comparing and Scaling*. The idea of ratio comparison was introduced there, along with informal ideas of equivalent ratios. These ideas are extended in the current unit. In *Moving Straight Ahead*, students will see proportional reasoning related to linear equations that pass through the origin.

In *Stretching and Shrinking*, the problem was finding dimensions of a larger (or smaller) physical or graphical model while preserving the relative size of the component parts so that the figures remained mathematically similar. The same ideas and ways of thinking developed in *Stretching and Shrinking* become powerful ways of thinking about ratios. The goal is the same in many ratio situations—to scale a pair of ratios up or down to determine whether they are equal.

A comparison problem in *Stretching and Shrinking* that called for finding the missing part of a ratio equivalent to a given ratio is the same as solving a proportion in *Comparing and Scaling*. For example, suppose you have a rectangle with dimensions of 5 cm by 7 cm. You want to draw a larger, similar, rectangle with the side corresponding to 5 cm being 15 cm. What would the other dimension be? This is an identical question: If roses are 5 for $7, how much will 15 roses cost? In each case, we are dealing with the given ratio of 5 to 7 and looking for the equivalent ratio of 15 to x. *Stretching and Shrinking* precedes *Comparing and Scaling* to give students experience with these ideas in a more concrete geometric context.

To summarize, the broad purposes of this unit are twofold. First, to develop students' ability to make intelligent comparisons of quantitative information using ratios, fractions, decimals, rates, unit rates, and percents. Second, to use quantitative comparison information to make larger or smaller scale models or scale rates and ratios up and down. An additional goal of this unit is to have students not only learn different ways to reason in proportional situations, but to recognize when such reasoning is appropriate.

Many important mathematical applications involve comparing quantities of one kind or another. In some cases, the problem is simply deciding which of two quantities is greater and describing how much greater it is. In such instances, we subtract to find a difference. This is what students deal with in elementary school. In fact, comparison by addition or subtraction comes first in students' mathematics experiences. This way of thinking becomes inappropriately pervasive in any situation requiring comparison.

Summary of Investigations

Investigation 1

Making Comparisons

Investigation 1 focuses on the language of comparisons and ratios in the context of advertising. Some content connects to questions asked of students in the sixth-grade work on percents and data analysis. Students learn what different kinds of comparative statements mean about the data that is given. They are asked to write comparative statements that describe data. Questions are asked that engage students in making comparisons and checking the accuracy of statements given. The important question of how you decide whether to use a difference, ratio, fraction, or percent to make a particular comparison is raised.

Investigation 2

Comparing Ratios, Percents, and Fractions

Investigation 2 builds on the variety of strategies for making comparisons—ratios, percents, and fractions—that arose in Investigation 1. The intent is to see how information in each of these forms provides the information needed to derive either of the other forms. Students investigate in more depth how ratios can be formed and scaled up or down to find equivalent ratios. This investigation more directly raises issues with comparison by finding differences.

Investigation 3

Comparing and Scaling Rates

Investigation 3 takes a specific focus on rates, scaling rates, and finding and interpreting unit rates as strategies. The investigation looks at scaling in numerical contexts; the connection to such proportional reasoning problems in geometry is made in Investigation 4. Rate tables are introduced as a tool for using scaling rates as a strategy for comparison. Students are asked to draw rate tables. They are also asked to write rules or equations. The ideas of average speed and constant speed are used. Students explicitly learn to use unit rates and to write equations and rules based on unit rates. Problem 4 confronts students with the need to label rates and unit rates carefully. When you divide to find a unit rate, determining what the division gives you is essential to making the comparison. Here students look at the measurement labels for assistance in determining what the quotient means.

Investigation 4

Making Sense of Proportions

Investigation 4 helps students write and use proportions to solve problems and make comparisons. All of the problems in this investigation can be posed in classical $\frac{a}{b} = \frac{x}{d}$ or $\frac{a}{b} = \frac{c}{x}$ form; yet, they are solved in a variety of equivalent ways. It is important that students learn different ways to reason in proportional situations, and recognize *when* such reasoning is appropriate. The strategies used to solve problems are based on students' knowledge of equivalent fractions. In one case, a geometric context ties to earlier work. In Problem 4.3, we look more systematically for an efficient strategy for solving proportions. We do not, however, cover cross multiplication. We have made a commitment to help students make sense of the strategies they use and feel that efficiency is only effective if students truly make sense of what they are doing. Therefore, we focus on scaling ratios up and down as a way of solving proportions. This builds on the substantial foundation for understanding and using equivalent fractions in the sixth-grade curriculum.

Mathematics Background

The subtitle of *Comparing and Scaling* is *Ratio, Proportion, and Percent*. This subtitle makes clear that the heart of the unit goals is to recognize when making comparisons using these strategies is appropriate, then to use these strategies with understanding and efficiency.

Scaling Ratios as a Strategy

To compare two or more related measures or counts, such as 3 roses for \$5 and 7 roses for \$9, you need strategies that allow the related pairs of numbers to be compared. Simple subtraction will not tell you what you want to know. Enter the world of ratio and proportion. A proportion is a statement of equality between two ratios. In this example, you need to find a way to scale the ratios of 3 to 5 and 7 to 9 so that they can be directly compared. Many students think these two ratios are the same, reasoning that 4 has been added to each of the numbers 3 and 5 to get 7 and 9. This is an example of students' misconceptions about when additive comparisons are appropriate. If you appropriately scale both ratios so that either the number of roses or the costs are the same, you are then left with a simple comparison of the quantities that are not the same. The two possibilities are shown below.

If you want to scale the costs to be the same, the kind of thinking is the same as that for finding a common denominator: look for a number that represents a multiple of the two numbers 5 and 9. If you scale to make the prices the same (that is, \$45), then the answer is immediately obvious.

$$\frac{3}{5} = \frac{3 \times 9}{5 \times 9} = \frac{27}{45} \text{ and } \frac{7}{9} = \frac{7 \times 5}{9 \times 5} = \frac{35}{45}$$

You can now compare the ratios 27 roses for \$45 and 35 roses for \$45. Clearly the second option gives more for the same amount of money.

Let's scale the numerators to be the same.

$$\frac{3}{5} = \frac{3 \times 7}{5 \times 7} = \frac{21}{35} \text{ and } \frac{7}{9} = \frac{7 \times 3}{9 \times 3} = \frac{21}{27}$$

You can now compare the ratios 21 roses for \$35 and 21 roses for \$27. Again the best buy is obvious.

This example underscores the relationship between the mathematical thinking used to find common denominators or common numerators in work with equivalent fractions and that was used to find equivalent ratios. Ratios are written in several forms. Some of the most often used are 2 to 3, or 2 : 3, or $\frac{2}{3}$. In the example, the convenience of writing the ratios as fractions helps the thinking needed for scaling the ratios up. However, we must make sure that students can differentiate between a ratio written as a fraction and a fraction representing the fractional part of a whole. We address this in the next section.

Using Ratio Statements to Find Fraction Statements of Comparison

The statement "the ratio of girls to boys in a class is 15 girls to 9 boys" can be written as the fraction $\frac{15}{9}$, but it does not mean that the fraction of students in the class that are girls is $\frac{15}{9}$. This is confusing for students and leads some teachers to avoid the fraction form for writing a ratio. We have chosen to confront the confusion by asking the fraction question directly.

> Maria says the fraction of the class that is girls is $\frac{15}{9}$. Bob says the fraction of the class that is girls is $\frac{15}{24}$. Who is correct and why?

The correct answer hinges on recognizing that a new quantity is actually used to find the fraction of students in the class that are girls. The total number of students in the class is needed. This is the sum of the numbers of boys and girls, 24. The part to whole comparison is $\frac{15}{24}$, and Bob is correct. Now we turn to another strategy for solving the roses problem.

Per Quantities: Finding and Using Rates and Unit Rates

If you compute the price per rose, you will have a rate comparison for the roses problem. In the 3 for \$5 deal, the unit rate is \$1.67. The price per rose in the 7 for \$9 deal is \$1.29—clearly the better price. Alternatively, at the 3 for \$5 price, 7 roses would cost \$11.67. This is a different comparison with the same result. Let's explore this strategy a bit further.

Here are two ratios that suggest rates:

> Sascha goes 5 miles in 20 minutes on the first part of his bike ride. On the second part, he goes 8 miles in 24 minutes. On which part is he riding faster?

Many students will intuitively want to divide the miles number and the minutes number to get a result, but they sometimes lose track of which one is divided into the other. Consequently they produce a number, but have no idea what the number means in the problem. Here the comparison can be made in two different ways by computing different unit rates. Let's look at each.

Suppose a student decided to divide 5 by 20 and 8 by 24. She gets the two numbers 0.25 and 0.333. What do these numbers mean? She might have divided 20 by 5 and 24 by 8. This division gives 4 and 3. What do these numbers mean? You have to know before you can decide what they tell us about the two legs of the bike ride. So start again and this time carry the label with the quantities.

$$\frac{5 \textit{ miles}}{20 \textit{ minutes}} = 0.25 \textit{ miles per minute} \text{ and}$$

$$\frac{8 \textit{ miles}}{24 \textit{ minutes}} = 0.333\ldots \textit{ miles per minute}$$

Now the comparison is clear. The times are the same, 1 minute, and the distances can be directly compared. 8 miles in 24 minutes is faster.

But, you could divide the other way:

$$\frac{20 \textit{ minutes}}{5 \textit{ miles}} = 4 \textit{ minutes per mile} \text{ and}$$

$$\frac{24 \textit{ minutes}}{8 \textit{ miles}} = 3 \textit{ minutes per mile}$$

Now you see that the lesser number tells the correct answer, 8 miles in 24 minutes.

What makes unit rates so interesting, and somewhat difficult, for students is that you have two options when you divide two numbers. The units help students think through such situations with the goal of building the flexibility to use either set of unit rates to compare the quantities.

One of the recurring themes of these materials is that we can represent data in different ways and that each way may tell us something that is not as obvious from other representations. The comparison in the rose example can be made in several ways: for example, using unit rates, comparing the ratios in fraction form to determine which is greater, or scaling both rates until the price is the same or the number of roses is the same. Developing strategies for deciding what the comparison situation calls for and for making comparisons are major goals of this unit.

Relating Ratios, Fractions and Percents

It is often desirable to change one form of comparison statement to another. The question is, can you write a percent statement given either a ratio or a fraction statement, and can you write a ratio or fraction statement given a percent comparison statement? Let's explore this with an example.

The ratio of concentrate to water in a mix for lemonade is 3 cups concentrate to 16 cups water. The questions you might ask are: "What fraction of the lemonade will be concentrate?" or "What percent of the lemonade will be concentrate?"

First find the total cups the recipe makes. It makes 19 cups. Then write the fraction of the lemonade that is concentrate, $\frac{3}{19}$. Now finding the percent is easy. Just divide the concentrate by the total, $3 \div 19 = 0.15789\ldots$ or about 15.8% concentrate.

Suppose you know that the percent of boys in a class is 48% and you want to write this as a ratio. You can think of the percent as a scaling of the ratio representing boys and girls up to a total of 100. So the girls are 52% of the class and the ratio of boys to girls is 48 to 52. You can scale this ratio down to 12 boys to 13 girls. The powerful thing about these related representations is the flexibility it gives us to choose the form of representation that describes the situation best for our purposes.

One caution about such changes of representation is that the choice to make these changes of form should be judged against whether the computations you do will have meaning. For example, in many rate situations, such as miles per gallon, trying to compute a percent does not make sense because the addition to get a total does not make sense. Miles covered plus gallons of gas used is a meaningless total. When the ratio can be thought of as part of a whole, the change of form we described makes sense (for example, white paint to blue paint in a mix, or high-fiber to high-protein nuggets in food for a baby chimp).

Proportions and Proportional Reasoning

The related concepts and skills in this unit are often referred to as *proportional reasoning*. Forming ratios in order to make comparisons is the heart of proportional reasoning. What is a proportion? A proportion is simply a statement of equality between two ratios. What makes this idea powerful is that if we know a ratio is equivalent to another, but we do not know both terms of one of the ratios, we can use what we already know about scaling or finding equivalent fractions to find the missing part of a proportion. Again, let's look at an example.

> It takes Glenda 70 steps on the elliptical machine to go 0.1 mile. When her workout is done, she has gone 3 miles. How many steps has she taken on the machine?

Here is a proportion and a solution for the number of steps that Glenda made.

$$\frac{70 \text{ steps}}{0.1 \text{ miles}} = \frac{x \text{ steps}}{3 \text{ miles}} = \frac{70 \times 30 \text{ steps}}{0.1 \times 30 \text{ miles}} = \frac{2{,}100 \text{ steps}}{3 \text{ miles}}$$

The first ratio in the proportion is scaled up by multiplying both the numerator and the denominator by 30. Thus, the denominator equals the denominator of the ratio with the unknown, x. This allows us to read the value of x directly since we know that if the two fractions are equivalent and have the same denominator, the numerators are also the same. The strategy we use to find the number by which we multiply, or "scale," is the same as the thinking process we use to find common denominators for fractions.

How far you go in formalizing the solving of proportions will depend on you and your students. We highly recommend that you do not impose solution strategies that have no meaning for the students. While cross multiplication is efficient, for most students at this level it is used without any understanding of why it works and consequently is often misused. We believe that students are better served by having the time to learn to think through situations requiring solving proportions and develop flexibility in approaching a problem so that easy possible solution strategies are not missed in a rush to an algorithm. This approach also builds on mathematics students already know and ways of thinking that they have already acquired. Helping students want to make sense of mathematics is encouraging a kind of thinking and flexibility that will allow them to feel confident to tackle problems that do not look exactly like ones they have already solved. Part of the goal of this unit is for students to learn to make judgments about the situation and to choose methods for comparing and for scaling.

Cross-Multiplying

If someone mentions cross-multiplication and the students seem interested, don't just give a procedure for cross-multiplication. Develop the idea based on what your students already know—finding common denominators. (If we use the product of the original denominators as a common denominator, the numerators will be cross products). In Question C part (2) of Problem 4.1, $\frac{7}{12} = \frac{x}{9}$, the common denominator will be 108. In the first fraction we have to multiply the numerator and denominator by 9. In the second fraction we need to multiply the numerator and the denominator by 12 to make the denominators the same.

Because the denominators are now the same, we need to find the value of x that makes the numerators equal. So we have to find x when $12x = 63$. This means that x must be 5.25. $\frac{7}{12} = \frac{x}{9}$ is equivalent to $\frac{63}{108} = \frac{12x}{108}$ is equivalent to $63 = 12x$, therefore $x = 5.25$.

So in a sense, cross-multiplying asks the question: What would be the numerator if these two fractions had a particular common denominator (the product of the original denominators)?

Helping students to make their own reasoning explicit can lead to a generalized method of solving proportions. For example, when many students solve this proportion $\frac{3}{7} = \frac{x}{343}$, they do the following arithmetic.

$$(343 \div 7) \times 3 = \frac{343}{7} \times 3 = \frac{343 \times 3}{7}$$

The division $343 \div 7$ finds the scaling factor by which we need to scale the 3.

Consequently, for solving a general proportion $\frac{a}{b} = \frac{x}{c}$, we can follow the same reasoning: find the scaling factor by computing $c \div b$, then multiply the scaling factor by a. So the arithmetic we actually perform is scale factor $\times$ the known numerator. In symbols this is $\frac{c}{b} \times a = x$ or $\frac{c \times a}{b} = x$.

With the unknown in the denominator, find the scale factor using the numerators so that we can scale the denominators to find the unknown. To solve $\frac{a}{b} = \frac{c}{x}$, we first find the scale factor by which we can make the numerators the same, $c \div a$. Then we have to scale the denominator to see what is equal to x. This gives $\frac{c}{a} \times b = x$ or $\frac{c \times b}{a} = x$.

An alternative strategy can be built using fact family ideas. To solve $\frac{a}{b} = \frac{c}{x}$, think of the equation as $\frac{a}{b} = c \div x$. From fact families, we can say that $x = c \div \frac{a}{b}$. Rewriting the right side with common denominators gives $x = \frac{cb}{b} \div \frac{a}{b} = cb \div a$, or $x = \frac{cb}{a}$.

These are the equations we would get by cross-multiplication, but here the explanation is built on students' ways of reasoning.

INTRODUCTION

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Content Connections to Other Units

Big Idea	Prior Work	Future Work
Ratio as a proportional relationship between quantities	Exploring and applying rational number concepts (*Bits and Pieces I, II, and III*)	Calculating and applying slope in equations of $y = mx + b$ form (*Moving Straight Ahead, Thinking With Mathematical Models, Say It With Symbols*)
Percent as a proportion that is always compared to 100	Percent defined as a ratio to 100 and connected to fractions and decimals (*Bits and Pieces I, II, and III*)	Making comparisons between groups of different sizes (*Data Around Us ©2004, Samples and Populations*)
Fractions as a ratio, rate, or as a part/whole relation	Fractions as a part/whole comparison; addition, subtraction, multiplication, and division with fractions (*Bits and Pieces I & II, How Likely Is It?*)	Expressing and applying probabilities as fractions (*What Do You Expect?*), determining if two algebraic expressions are equivalent (*Say It With Symbols*)
Scaling to determine one quantity in terms of another and to find equivalent ratios	Comparing and subdividing similar figures to determine scale factors (*Stretching and Shrinking*)	Scaling up rectangular prisms (*Filling and Wrapping*)
Comparing quantities using ratios, proportions, rates, or percents	Connecting and comparing rates using ratios, decimals, and percents (*Bits and Pieces I & II*), comparing data sets (*Data About Us*)	Comparing probabilities (*What Do You Expect?, Samples and Populations*), comparing data sets (*Data Around Us ©2004*)
Developing strategies and techniques to solve for missing values in a proportion	Making inferences about quantities and populations based on experimental or theoretical probabilities (*How Likely Is It?*)	Developing benchmarks and skills for estimating irrational numbers (*Looking for Pythagoras*), estimating populations (*Samples and Populations*), estimating with and comparing large numbers (*Data Around Us ©2004*)

Overview

In the middle grades, students are introduced to fractions and decimals. The next major hurdle is building an understanding of positive and negative numbers, i.e., integers, fractions, and decimals. Students have experienced these kinds of numbers informally in their everyday world. For example, temperatures drop below zero in the winter or soar above 90 degrees in the summer, and sports teams are said to be ahead or behind by so much. Students have intuitively used operations on integers to make sense of these situations. This unit explores situations that require representation with positive and negative numbers. These situations motivate more formal ways to add, subtract, multiply, and divide these numbers. Students formalize algorithms for operating using positive and negative numbers. They also consider the order of operations and selected properties.

Summary of Investigations

Investigation 1

Extending the Number System

This investigation gives students experiences with positive and negative numbers, ordering, and informal operations in a variety of contexts so that subsequent formal work can be based on "what makes sense." Positive and negative numbers in the form of integers, fractions, and decimals are also represented on a number line.

Investigation 2

Adding and Subtracting Integers

Students build on the informal work of Investigation 1 to formulate algorithms for addition and subtraction of positive and negative numbers. In each problem, students are encouraged to think about the meaning of the operations from several perspectives and use different representation models (number line and chip board).

Investigation 3

Multiplying and Dividing Integers

This investigation is structured and developed in a style parallel to that of Investigation 2. The number line model and fact families as well as the contexts of time, distance, and speed are used to develop students' understanding of multiplication and division of positive and negative numbers. Since students have not done much informal multiplication and division, a useful context for questions leading to multiplication and division is explained before asking students to formulate algorithms. The context involves motion at various rates in both directions on a number line. The cases of a negative number times a positive number and a negative number divided by a positive number could come quite easily from the chip board context. However, that model doesn't seem to lead naturally to those cases that involve the product or quotient of two negatives. In almost any context, you have to think hard to get a reasonable guide to the operations we want to develop. In the time, rate, distance, and position setting, these ideas are plausible. The cases of combinations of "signs" for multiplication are explained by looking at number patterns.

Investigation 4

Properties of Operations

Students compare algebraic properties of the operations on positive and negative numbers (i.e. the rational numbers) to those of the number system of only positive numbers (whole numbers). It's not intended to be a full-scale treatment of field properties of the real numbers.

Mathematics Background

Most students may be able to add, subtract, multiply, and divide whole numbers, fractions, and decimals. However, most have not been asked to think about what the operations mean and what kinds of situations call for which operation. Students need the development of the disposition to seek ways of making sense of mathematical ideas and skills. Otherwise, they may end up with technical skills but without ways of deciding when and how those skills can be used to solve problems.

One way to develop the desire to make sense of these ideas is to model such thinking in classroom conversation. Asking questions about meaning, (about what makes sense) as a regular, expected part of classroom discourse helps focus students on making connections. Exploring new aspects of numbers in a way that builds on and connects to what they already know is likely to have two good effects. First, students will deepen their understanding of familiar numbers and operations. Second, the new numbers, integers, and negative rational numbers will be more deeply integrated into students' own mathematical knowledge and resources.

Students find several things difficult about working with positive and negative numbers.

- The fact that -27 is less than -12 is contrary to students' experience with whole numbers (positive integers and zero). This understanding requires building mental images and models that allow students to visualize these new comparisons and relationships.
- The operation of subtraction, especially of subtracting a negative number, is difficult for students to understand. In this unit, students will have several opportunities to think about what makes sense and why. They will encounter representations and models that will help them better understand subtraction.
- The idea that subtracting a negative number gives the same result as adding the opposite of the negative number (adding a positive) is difficult for many students. This understanding must develop over time as students make observations and comparisons between subtraction and addition. Recognizing that these are inverse operations and that addition sentences are related to subtraction sentences helps students to expand their understanding of this concept.
- Multiplying two negatives and getting a positive number does not make sense to many students. In fact, the usual ways of giving meaning to multiplication, such as repeatedly adding an amount, seem of little help in making sense of $-12 \times (-5)$.
- In the unit, we approach these difficult concepts of subtracting a negative and multiplying two negatives through the use of set and number line models and the relationships that exist between addition and subtraction and multiplication and division.
- A number of other confusions occur. For example, the idea that a negative rational number names a point on the number line or knowing that $-3\frac{1}{2}$ can be thought of as $-3 + (-\frac{1}{2})$ is often not transparent.

Using Models for Integers and the Operations of Addition and Subtraction

The number line is a model that is used throughout the number units. It was first introduced in *Bits and Pieces I* to develop understanding of equivalence of fractions and decimals. It was used in *Bits and Pieces II and III* to help develop the operations for fractions and decimals. It is used later in *Looking for Pythagoras* to introduce square roots and irrational numbers. In this unit, students use the directed distance model with the number line to visualize adding and subtracting integers. Here are two situations that students encounter that use both *distance* and *direction* as ways to consider integers.

> *The world record for fastest rise in outside air temperature occurred in Spearfish, South Dakota, on January 22, 1943. The temperature rose from $^{-}4$°F to $^{+}45$°F in two minutes. What was the change in temperature over those two minutes?*

On a number line, this change can be shown using an arrow. (Figure 1)

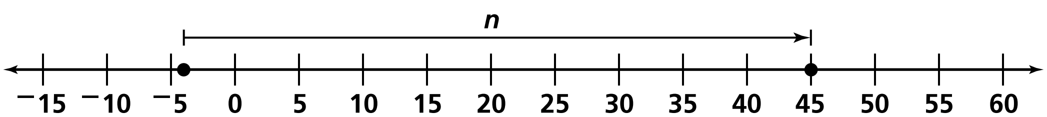
A student might reason: "From $^{-}4$°F to 0°F is an increase of $^{+}4$°F, and from 0°F to 45°F is an increase of $^{+}45$°F. So the total change is an increase of $^{+}49$°F." The situation can be described with a number sentence:

$$^{-}4° + n = {}^{+}45° \quad \text{or} \quad {}^{-}4° + {}^{+}49° = {}^{+}45°$$

The sign of the change in temperature shows the direction of the change. If the temperature had dropped 10°F, the student would write the change as $^{-}10$°F to show the size and direction of the change. (Figure 2)

$$^{-}4° + n = {}^{-}14° \quad \text{or} \quad {}^{-}4° + {}^{-}10° = {}^{-}14°$$

Figure 1



We can write these equations without the degree markers. We just have to remember what the answer means. To facilitate the development of the algorithms, the absolute value concept is introduced in Investigation 2 as a way to talk about distance on the number line. It also helps to talk about the value of a number when direction is not considered.

Colored chips can also be used to develop a strategy for adding and subtracting integers. Using this model requires an understanding of opposites. For example, $^{-}3$ and $^{+}3$ are opposite because $^{+}3 + ^{-}3 = 0$, or each number is equidistant from the origin on the number line. Red-black pairs represent opposites ($^{-}1$ and $^{+}1$), which add to 0. The chip model uses one color of chips (black) to represent positive integers and another color (red) to represent negative integers. (**Note:** You may use any collection of two-color chips—designate which color is positive and which color is negative.)

To use the model with addition, begin with an empty chip board. Place chips on the board to represent each addend. If the integer is positive, place that number of black chips on the board. If the integer is negative, place that number of red chips on the board. If the two integers being added have the same sign, the sum is the total number of chips on the board. For example, to add $^{-}4 + ^{-}3$, place 4 red chips and then another 3 red chips on the board for a total of 7 red chips (representing a sum of $^{-}7$, or $^{-}4 + ^{-}3 = ^{-}7$).

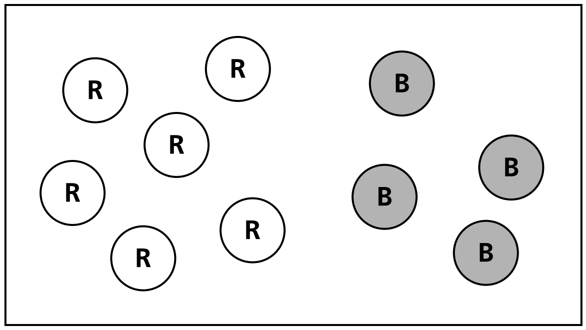
If the integers being added have different signs, place the appropriate number of red and black chips on the board to represent each addend. Simplify the board by removing red-black (opposite) pairs of chips. The chips that remain unmatched represent the sum of the two integers.

Consider this problem:

Linda owes her sister $6 for helping her cut the lawn. She earns $4 delivering papers with her brother. Is she "in the red" or "in the black"?

Using collections of black and red chips on a chip board, you can represent the combination of expense and income.

Chip Board



The result, or net worth, is that Linda is "in the red" 2 dollars, or $^{-}2$ dollars. This problem may be represented with the number sentence, $^{-}6 + ^{+}4 = ^{-}2$.

Because each chip represents 1 unit, either positive or negative, red and black chips are thought of as opposites. Combining two opposite chips makes zero ($^{+}1 + ^{-}1 = 0$). In this problem, we can rewrite $^{-}6$ as $^{-}2 + ^{-}4$ so that 4 chips of each color can be paired to make zeros ($^{-}6 + ^{+}4 = ^{-}2 + ^{-}4 + ^{+}4 = ^{-}2 + 0$). After the paired chips are removed, 2 red chips remain. These chips represent the sum $^{-}2$ ($^{-}2 + 0 = ^{-}2$).

Here is another problem that can be modeled and solved using chips:

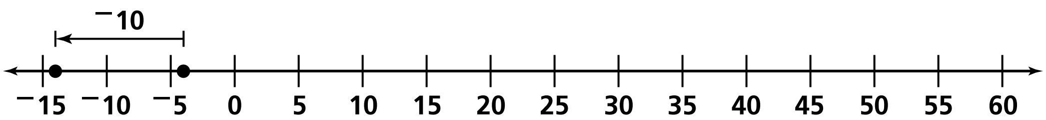
Jeremy earns $10 mowing a lawn. He needs to pay $15 to rent his equipment. How much more money does he need to pay the rental cost?

This problem may be modeled using chips by representing the $10 with a combination of 15 black and 5 red chips ($10 = ^{-}5 + 15$). Now $15 or 15 black chips can be "taken away" [$^{-}5 + (15 - 15)$], leaving 5 red chips to represent the $^{-}5$ that Jeremy is "short." Two different number sentences are applicable:

$^{+}10 + ^{-}15 = ^{-}5$ and $^{+}10 - ^{+}15 = ^{-}5$

As students work on adding and subtracting integers, they notice that it may be helpful to restate an addition problem as a subtraction

Figure 2



problem or vice versa. This involves using opposites of numbers. For example:

To calculate $^{+}12 + {}^{-}8$, they may realize that the result is the same as when they subtract $^{+}8$ in the problem $^{+}12 - {}^{+}8$.

For $^{+}5 - {}^{-}7$, they may realize that the result is the same as when they add $^{+}7$ in $^{+}5 + {}^{+}7$.

Students build on generalizations made using models of integers to extend their work with negative and positive rational numbers.

Fact Families

Fact families are used in this unit to help students understand the relationship between addition and subtraction and between multiplication and division. Fact families were introduced in the number units in grade 6. Here is an example of a fact family:

$^{-}7 + {}^{+}2 = {}^{-}5$ $\quad$ $^{-}5 - {}^{+}2 = {}^{-}7$

$^{+}2 + {}^{-}7 = {}^{-}5$ $\quad$ $^{-}5 - {}^{-}7 = {}^{+}2$

Fact families are also used to find a missing factor or addend such as:

$^{+}4 + n = {}^{+}43$ or $^{-}6n = 42$

Models and the Operations of Multiplication and Division

Multiplication can be explored using a number line model and "counting" occurrences of fixed-size movement along the number line. Direction of movement introduces negative and positive movements. For example:

Hahn passes the 0 point running 5 meters per second to the right. Where is he 10 seconds later?

Aurelia passes the 0 point running to the left at 6 meters per second. Where is she 8 seconds later?

Relating division to multiplication helps develop division with integers. A multiplication fact can be used as the basis for creating two related division facts. By developing division this way, students can determine the sign of the answer to a division problem. For example, since we know that $5 \times ({}^{-}2) = {}^{-}10$ (or $^{-}2 \times 5 = {}^{-}10$), we can write related division sentences:

$^{-}10 \div ({}^{-}2) = 5$ $\quad$ and $\quad$ $^{-}10 \div 5 = {}^{-}2$

Then, students can generalize rules for handling the sign of the quotient in a division problem.

Some Notes on Notation

Writing integers with raised signs avoids confusion with symbols for addition and subtraction. However, most software and most writing in mathematics do not use the raised signs.

Positive numbers are usually written without a sign.

$^{+}3 = 3$ and $^{+}7.5 = 7.5$.

Negative numbers are usually written with a dash like a subtraction sign.

$^{-}3 = -3$ and $^{-}7.5 = -7.5$.

Parentheses can help.

$^{-}5 - {}^{-}8 = {}^{-}5 - {}^{-}8 = {}^{-}5 - ({}^{-}8)$

The subtraction symbol also indicates the opposite of a number. For example, $^{-}8$ represents the opposite of 8 and $^{-}({}^{-}8)$ the opposite of $^{-}8$.

$^{-}({}^{-}8) = 8$

We use raised signs for the first two investigations, after which we use the standard notation.

For multiplication, you can use a raised dot symbol. For example, $3 \times 5 = 3 \cdot 5$. Some students might have seen $3 \cdot (4 + 5)$ or $3 \times (4 + 5)$, or even $3(4 + 5)$.

Order of Operations and Properties

Order of operations rules are introduced.

1. Compute any expressions within parentheses.

   $({}^{-}7 - 2) + 1 = {}^{-}9 + 1 = {}^{-}8$

   $(1 + 2) \times (-4) = 3 \times ({}^{-}4) = {}^{-}12$

2. Compute any exponents.

   $^{-}2 + 3^2 = {}^{-}2 + 9 = 7$

   $6 - ({}^{-}1 + 4)^2 = 6 - (3)^2 = {}^{-}3$

3. Multiply and divide in order, from left to right.

   Example 1

$1 + 2 \times 4 =$	Multiplication first
$1 + 8 = 9$	

   Example 2

$200 \div 10 \times 2 =$	Division first
$20 \times 2 = 40$	Multiplication second

4. Add and subtract in order, from left to right.

$1 - 2 + 3 \times 4 =$	Multiplication first
$1 - 2 + 12 =$	Addition and subtraction
$^{-}1 + 12 = 11$	

The Commutative Property of Addition and Multiplication is introduced. Students find that this property does not hold for subtraction or division of integers. The Distributive Property of Multiplication over Addition or Subtraction is also introduced and modeled through finding areas of rectangles. The Associative Property is explored in an ACE exercise. These properties are revisited in several succeeding units, particularly the algebra units.

Content Connections to Other Units

Big Idea	Prior Work	Future Work
Defining and developing understanding of positive and negative numbers	Developing understanding of whole numbers and rational numbers (*Prime Time, Bits and Pieces I, II, & III*)	Interpreting and applying positive and negative slopes of lines and positive and negative coefficients in equations (*Moving Straight Ahead, Thinking With Mathematical Models, Say It With Symbols, The Shapes of Algebra*); developing understanding of square roots and irrational numbers (*Looking for Pythagoras*)
Exploring relationships between positive and negative numbers (e.g., interpreting positive numbers as a loss and negative numbers as a gain)	Using models to develop understanding of mathematical concepts (*Bits and Pieces I, II, & III, Covering and Surrounding, Ruins of Montarek, Stretching and Shrinking, Comparing and Scaling*)	Understanding relationships between positive and negative coefficients or values for variables (*Moving Straight Ahead, Thinking With Mathematical Models, Say It With Symbols, The Shapes of Algebra*); using positive and negative integers to communicate directions in two dimensions (*Kaleidoscopes, Hubcaps, and Mirrors*)
Developing understanding of arithmetic operations with positive and negative numbers	Understanding and applying arithmetic operations with rational numbers (*Bits and Pieces II & III, Stretching and Shrinking, Comparing and Scaling*)	Evaluating algebraic expressions involving positive and negative coefficients or values for variables (*Moving Straight Ahead, Data Distributions, Thinking With Mathematical Models; Frogs, Fleas, and Painted Cubes; Say It With Symbols, The Shapes of Algebra, Clever Counting* ©2004); interpreting isometries in the plane given in symbolic form (*Kaleidoscopes, Hubcaps, and Mirrors*)
Extending the number line and coordinate grid to include negative coordinates	Using a coordinate grid with positive coordinates (*Data About Us, Covering and Surrounding, Variables and Patterns, Stretching and Shrinking*); using a number line to develop equivalence and operations of fractions and decimals (*Bits I, II, and III*)	Graphing equations on coordinate grids (*Moving Straight Ahead, Data Distributions, Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols, The Shapes of Algebra; Kaleidoscopes, Hubcaps, and Mirrors*); locating square roots on the number line (*Looking for Pythagoras*)
Developing understanding of the Commutative Property, the Distributive Property, and the order of operations	Developing understanding of the Commutative Property using whole numbers and rational numbers (*Prime Time, Bits and Pieces II & III*); using the order of operations to solve problems in a context (*Covering and Surrounding, Variables and Patterns*)	Using the properties and order of operations to write equivalent expressions and solve equations (*Moving Straight Ahead, Thinking With Mathematical Models; Growing, Growing, Growing; Frogs, Fleas, and Painted Cubes; Say It With Symbols, The Shapes of Algebra*)