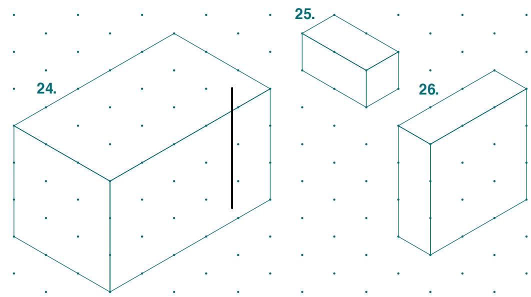
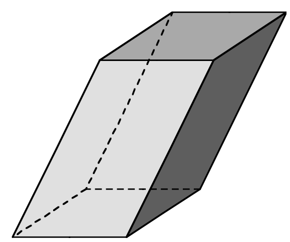
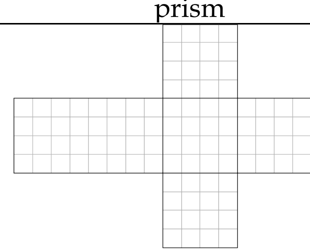
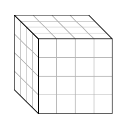
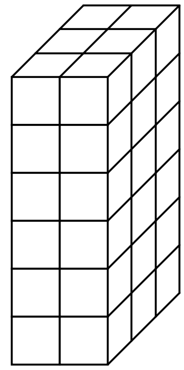
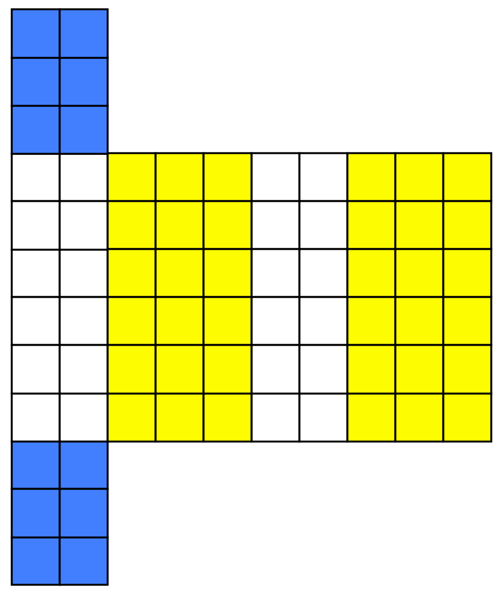
Vocabulary: *Filling and Wrapping*

Concept	Example
A rectangular prism : is a three-dimensional shape with congruent rectangles for **base and top,** and four **lateral (side)** rectangular faces. (Technically, this defines a **right rectangular prism**. If the lateral faces were parallelograms then this would be an **oblique prism**. Students do not investigate oblique prisms in this unit.) Note: In a rectangular prism, front and back faces are congruent, top and bottom faces are congruent, right and left sides are congruent.	 Above are some **right rectangular prisms.** The "right" refers to the angle between faces. The **base** of the box on the left is a 3 by 5 rectangle, so is the top. Two of the **lateral** faces of this box are 3 by 5 rectangles; the other two faces are 3 by 3 rectangles.  Oblique rectangular prism
Surface area of a Rectangular Prism: This is the area needed to "wrap" the prism, that is, the sum of the areas of the 6 faces. A **pattern** is a flat design, that is, a 2-dimensional shape that can be folded to make or wrap a rectangular prism. This provides a visual representation of surface area as a two-dimensional measure of a three-dimensional object. This also provides a strategy to find the surface area of a prism. **Surface area** = area of (base + top + front + back + left side + right side) = 2(base) + 2(front) + 2(right side)	  The above is **a pattern** for a 4 by 4 by 4 **cube.** A cube is a special case of a rectangular prism. The bases and the lateral faces are all squares, which are just special cases of rectangles. *Find the surface area of the rectangular prism shown below.*

= 2(length x width) + 2(length by height) + 2(width x height).



The base and top are 2 by 3 rectangles. The front and back are 2 by 6. The sides are 3 by 6. The **dimensions,** length by width by height, are 2 by 3 by 6.

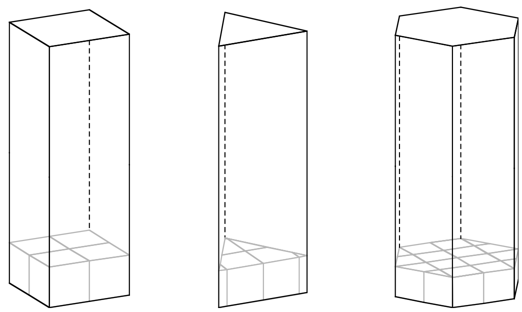


This **pattern** would fold up to make the prism pictured above. The white 2 by 6 rectangles would cover the front and back. The yellow 3 by 6 rectangles cover the right and left faces. The blue 2 by 3 rectangles cover the top and bottom. Total surface area = 2(2 x 6) + 2(3 x 6) + 2(2 x 3) = 72 square units.

Note: other patterns could have been drawn to show how the 6 faces might be laid out, but the surface area would not be affected. The prism could also be reoriented so that another face is on the bottom. It does not matter which face we call the base, since they are all rectangles.

General Prisms: The definition of a right rectangular prism generalizes to fit all other prisms. A **prism** is a three-dimensional shape with congruent polygons for **base and top** and rectangles for **lateral** faces. Thus, a **pentagonal prism** has a polygon for a base and top and 5 rectangular faces for the lateral faces. A **hexagonal prism** has a hexagon for a base and top and 6 rectangular lateral faces. If the base polygon is regular then the lateral faces will be congruent rectangles, but the base polygon does not have to be regular.

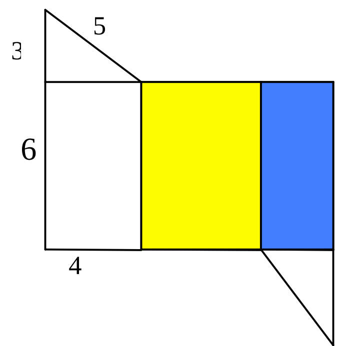
These are all prisms.



Rectangular Triangular Pentagonal

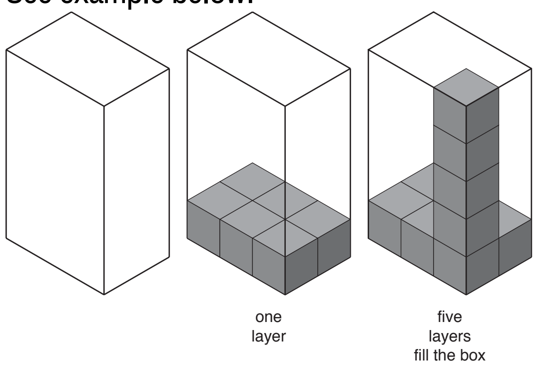
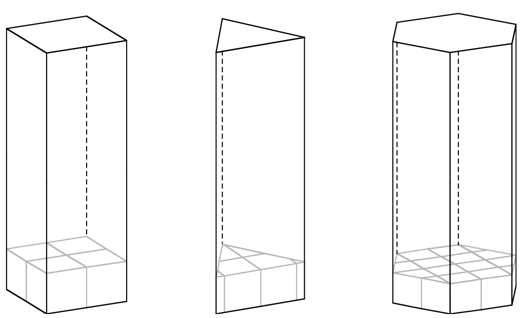
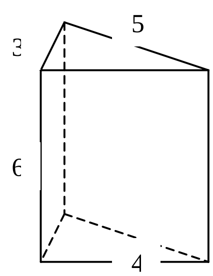
Notice that the lateral faces are congruent rectangles in the above prisms, because the each base is a regular polygon.

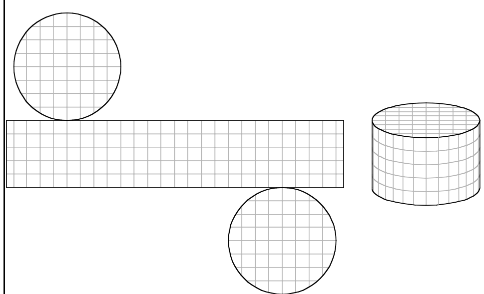
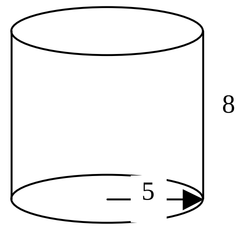
Find the surface area of a triangular prism whose base is a right triangle with sides 3, 4 and 5 units, and whose height is 6 units.

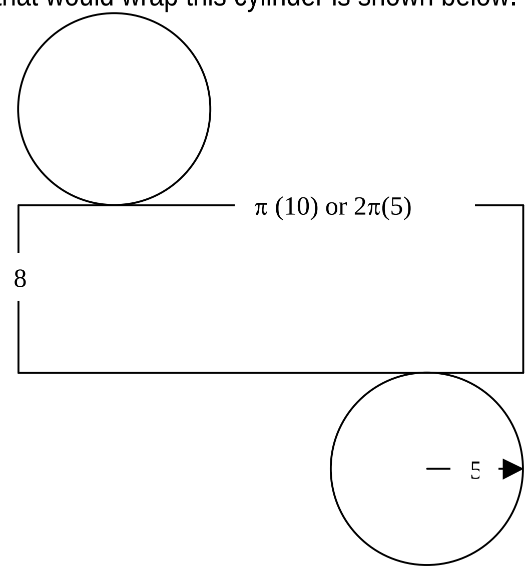


We can take advantage of the known sides of the triangle to deduce the dimensions of the yellow rectangle, 5 by 6, and the blue rectangle, 3 by 6. These are not congruent rectangles because the base is not regular.

Surface area = 2 triangles + 3 rectangles = 2(0.5 x 4 x 3) + 4 x 6 + 5 x 6 + 3 x 6 = 2(6) + 24 + 30 + 18 = 84 square units.

The **volume of a rectangular box or prism**: the number of unit cubes it takes to fill the box or prism. One strategy to determine this is to count the number of unit cubes in the layer that would cover the area of the base—one unit cube sits on each square unit in the base -- and to count the number of layers needed to equal the height of the prism. The **volume (the total number of unit cubes) of a rectangular prism, therefore, is the area of its base x height. See example below:**  	*Find the volume of a rectangular prism with dimensions 2 by 5 by 8.* We can consider any of the faces to be the base. Suppose we decide that the base is 2 by 5 units, then the height is 8 units. The base layer would contain 2 x 5 or 10 unit cubes, and there would be 8 layers. Thus, the volume is 8 x 10 cubic units. *Find the volume of a rectangular prism with dimensions 2.5 by 5 by 3.3.* Volume = (area of base) x height = (2.5 x 5) x 3.3 = 12.5 x 3.3 = 41.25 cubic units.
The **volume of any prism = base area x height.**  The figures indicate how the layering strategy generalizes to any kind of prism.	*Find the volume of a right triangular prism whose base is a 3 by 4 by 5 right triangle, and whose height is 6 units.*  Since the triangle is a right triangle we know its base and height, 4 units and 3 units. Thus, the base area = 0.5(4 x 3) = 6 square units. The height of the prism is 6 units. Therefore, the volume of the prism = base area x height = 6 x 6 = 36 cubic units.
Cylinder: is a particular kind of variation of a prism. The base and top are congruent circles, not polygons, and the lateral face is one continuous rectangle. Therefore, the strategies for finding the surface area and volume of a prism extend to finding the	*Find the surface area and volume of the cylinder shown below.*

<table>
<tr>
<td>surface area and volume of a cylinder.
Surface area of cylinder = area of base and top + lateral area
= 2(area of circular base) + area of lateral rectangle.
Because the lateral rectangle wraps around the base circle we know that the length of this lateral rectangle = circumference of base circle = **π**(diameter).
Thus, surface area = 2(**π**r^2) + **π**d

The volume of a cylinder is the number of unit cubes in one layer (the area of the circular base) multiplied by the number of layers (the height) needed to fill the cylinder. Thus,
Volume = (**π**r^2)(h) cubic units.</td>
<td>

The radius is 5, so the diameter is 10 units. A pattern that would wrap this cylinder is shown below.


Surface area = 2(circles) + 1 rectangle
= 2(πr^2)+ π(d)(h) = 2(3.14)(25) + 3.14(10)(8) = 408.4 square units. (approx).
Note: If we use the approximation 3.14 for π then we have to indicate that the answer is accurate but not exact.

Volume = area of base layer x number of layers
= (πr^2)(height) = 3.14(25)(8) = 628 cubic units.</td>
</tr>
<tr>
<td>Cone: a three dimensional shape with a circular base, which rises to a single vertex. Since the base and top are not congruent this is not a prism or a cylinder. By experimenting students compare the volumes of a cylinder and a cone, where they each have the same radius of base, and same height. They discover it takes 3 cones to fill the cylinder. Therefore,</td>
<td>*Find the volume of a cone which has a base with diameter 10 units and height 8 units.*</td>
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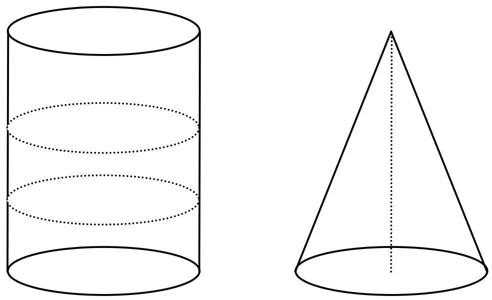
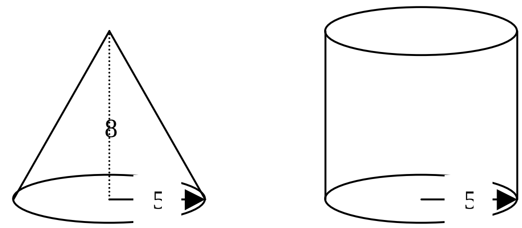
<tr>
<td>volume of the cone = $\frac{1}{3}$ volume of the cylinder with same base and height.



Thus, volume of cone = $\frac{1}{3}$(**π**r^2)(h) cubic units.

Note: The surface area of a cone is not part of this curriculum.</td>
<td>

The cylinder is drawn for comparison. We know the volume of the cylinder = **π**r^2(h) = 3.14(25)(8) = 628 cubic units (approx).

Therefore, the volume of the cone = ($\frac{1}{3}$)(628) = 209.3 cubic units (approx).</td>
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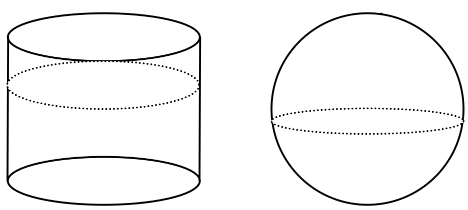
<tr>
<td>Sphere: By experimenting students compare the volumes of a cylinder and a sphere, where they each have the same radius, and same height.



Volume of the sphere = $\frac{2}{3}$ of the volume of the cylinder with same radius and height.

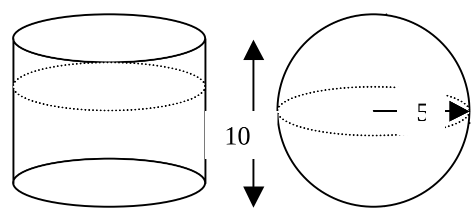
Thus,

Volume of sphere = $\frac{2}{3}$(**π**r^2)(h) cubic units.

Note: height of sphere = diameter of sphere or 2r, so the formula can be rewritten as:

Volume of sphere = $\frac{2}{3}$(**π**r^2)(2r)

= $\frac{4}{3}$ (**π**r^3) cubic units.</td>
<td>*Find the volume of a sphere with radius 5 units.*



The cylinder is drawn for comparison. We know the volume of the cylinder = area of base x height = (**π**r^2)(h) = 3.14(25)(10) = 785 cubic units (approx).
Therefore, the volume of the sphere = ($\frac{2}{3}$)(785) = 523.3 cubic units.</td>
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<table>
<tr>
<td>Note: Surface area of a sphere is not part of this curriculum.</td>
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<tr>
<td>Relationship between surface area and fixed volume for rectangular prisms: As students discovered in Covering and Surrounding, a fixed area can be surrounded by different perimeters, and the most efficient rectangular perimeter is a square.
Analogous to this idea is that a fixed volume can be “wrapped” by different surface areas, and the most efficient rectangular prism is a cube (However, if we are permitted to use a cylinder or a sphere then we can package the same fixed volume into less wrapping.)</td>
<td>Find at least two different rectangular prisms with a volume of 36 cubic units.

Since volume = base area x height we need factor pairs of 36, one factor for the base and one factor for the height.
<table>
<tr><td>Base area</td><td>Height</td><td>volume</td></tr>
<tr><td>1</td><td>36</td><td>36</td></tr>
<tr><td>2</td><td>18</td><td>36</td></tr>
<tr><td>3</td><td>12</td><td>36</td></tr>
<tr><td>4</td><td>9</td><td>36</td></tr>
<tr><td>6</td><td>6</td><td>36</td></tr>
</table>
For each of these base areas we have alternatives for the length and width. For example, if we investigate a base area of 4 then this might be achieved with a length of 1 and width of 4, or a length of 2 and a width of 2:
<table>
<tr><td>Length</td><td>Width</td><td>Height</td><td>Volume</td></tr>
<tr><td>1</td><td>4</td><td>9</td><td>36</td></tr>
<tr><td>2</td><td>2</td><td>9</td><td>36</td></tr>
</table>

These two rectangular prisms are pictured below.</td>
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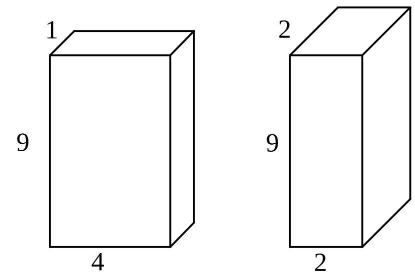


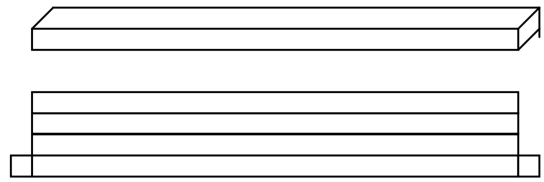
Find the rectangular prism with the least surface area that holds a volume of 36 cubic units.

Expanding the table of length and width above we have:

Length	Width	Height	Volume	Surface Area
1	1	36	36	?
1	2	18	36	?
1	3	12	36	?
1	4	9	36	?
1	6	6	36	?
2	2	9	36	?
2	3	6	36	?
3	3	4	36	?

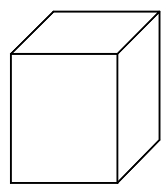
Note: these are only the whole number dimensions. If we are permitted to investigate fractional dimensions we would have an infinitely long table.

Students discovered that the longest and thinnest packages were least efficient in their use of surface area to contain a fixed volume, while those rectangular prisms that most resemble a cube are most efficient. Thus, the least surface area will be used by the 3 by 3 by 4 box, while the most will be used by a 1 by 1 by 36 box. To confirm this, we can calculate the surface area of each of these.

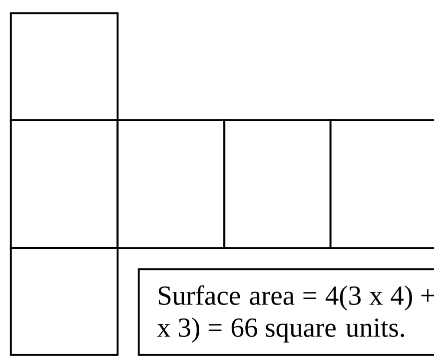


Sketch of 1 by 1 by 36 rectangular prism and the pattern that would wrap it.

Surface area = 2(1 x 1) + 4(1 x 36) = 146 square units.



Sketch of 3 by 3 by 4 prism and pattern to wrap it.



Surface area = 4(3 x 4) + 2(3 x 3) = 66 square units.

Find a cylinder that will hold the same volume as a 3 by 3 by 4 rectangular prism. How do the surface areas compare?

The 3 by 3 by 4 rectangular prism has a volume of 36 cubic units. The surface area is 66 square units (see previous example). For a cylinder to have a volume of 36 cubic units we need to have base area x height = 36. Therefore, we have an infinite number of factor pairs to consider. If we think only of whole number factors then we have:

Base area	Height	Volume
1	36	36
2	18	36
3	12	36
4	9	36
6	6	36
9	4	36
12	3	36
18	2	36
36	1	36

If the base area is 1 square unit and the height is 36 then the cylinder is long and thin. If the base area is 36 and the height is 1 the cylinder is short and wide. Following are calculations for
Surface area of 3 of these cylinders, height 36, height 3, height 1.

a. Height 36, base 1 square unit. The base is a circle with area = πr^2 = 1. So 3.14(r^2) = 1, so r = 0.564 units. The circumference of this circle = π(1.128) = 3.54 (approx). The surface area of this cylinder = 2 base circles + lateral rectangle = 2(1) + (3.54)(36) = 129.44 square units.
b. Height 3, base 12 square units. The base is a circle with area = πr^2 = 12. So 3.14(r^2) = 12, so r = 1.955 units. The circumference of this circle = π(3.91) = 12.28 units. The surface area of this cylinder = 2 base circles + lateral rectangle = 2(12) + (12.28)(3) = 60.84.
c. Height 1, base 36 square units. The base is a circle with area = πr^2 = 36. So r = 3.39 units. The circumference = πd = 3.14(6.77) = 21.26 units. The surface area = 2 circles + lateral rectangle = 2(36) + (21.26)(1) = 93.16 square units.

If we are interested in wrapping a volume of 36 cubic units and can use either a cylinder or a 3 by 3 by 4 rectangular prism, then the cylinder with height 3 and base 12 uses less surface area than the rectangular prism, and the cylinder s with height 1 or height 36 use more surface area.

Effect of changing 1, 2 or 3 of the	*How are the volume and surface area of a 2 by 3 by*

dimensions of a rectangular prism: If every dimension of a rectangular prism is changed by the same scale factor, k, (See *Comparing and Scaling*) then the new prism created is similar to the original prism and the volume will changed by a factor of k^3.	*5 box changed if we* *a. Double only the length or* *b. Double each of the dimensions?* a. The volume of the original 2 by 3 by 5 box is 30 cubic units. The surface area is 2(2 x 3) + 2(2 x 5) + 2(3 x 5) = 62 square units. If we double only the length then the box has dimensions 4 by 3 by 5; the volume of this is 60 cubic units and the surface area = 2(4 x 3) + 2(4 x 5) + 2(3 x 5) = 94 square units. That is, the volume has been doubled but the surface area has not. b. If we double all the dimensions then the box is now 4 by 6 by 10; the volume of this is 240 cubic units and the surface area = 2(4 x 6) + 2(4 x 10) + 2(6 x 10) = 248 square units. That is, the volume has been increased by a factor of 8 (which is 2^3) and the surface area has been increased by a factor of 4 (which is 2^2).