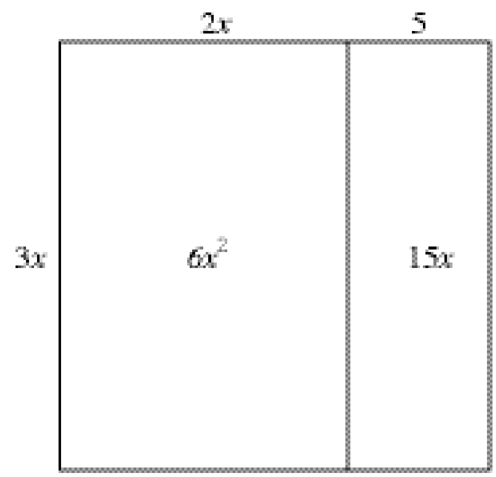
Vocabulary: Frogs, Fleas and Painted Cubes

Concept	Example
Equation of a quadratic function or relationship: fits the form $y = a\mathbf{x^2} + bx + c$, in which *a*, *b*, and *c* are constants, $a \neq 0$. There must be an "x-**squared**" term, and this must be the highest power in the equation. Note: the presence of an x-squared term indicates that some multiplication with the independent variable x has taken place to create this equation. This hints at the type of contexts to which quadratic relationships apply. **Expanded form of the equation**: is $y = ax^2 + bx + c$ for some values of a, b, c where $a \neq 0$. **Factored form of the equation**: is equivalent to the above but has the form $y = (ax + c)(bx + d)$, for some values of *a, b, c, d,* where $a \neq 0$ and $b \neq 0$. "ax + c" is one factor and "bx + d" is another factor. The two factors are multiplied. Note: it is *not* always possible to start with a quadratic in expanded form and rewrite in factored form, with rational values for *a, b, c, d*. It *is* always possible to start with a quadratic equation in factored form and rewrite in expanded form. **The distributive property and equivalent quadratic expressions:** The **distributive property for multiplication** over addition states that a(x+ c) = ax + ac; for example, 3(2 + 10) = 2(2) + 3(10). This can be expanded to multiplying (ax + b)(cx + d). y = (ax + b)(cx + d) is the **factored form** of	1. *Use the **distributive property** to multiply 3(2x + 5) and 3x(2x + 5). Which is a **quadratic** expression?* 3(2x + 5) = 3(2x) + 3(5) = 6x + 15. This is a *linear* expression (See *Moving Straight Ahead*), since there is no power higher than 1. (Note: Students first met the distributive property in *Accentuate The Negative.*) One of the ways students can make sense of this property is to picture it as an area model, where one *factor* is the width of a rectangle and the other is the length.  Therefore, $3x(2x + 5) = 6x^2 + 15x$. The presence of the "$6x^2$" indicates this is a **quadratic** expression. 2. *Use the **distributive property** to write the **factored form**, y = (x + 2)(x + 5), in **expanded** form.*

a quadratic. If the distributive property is applied to this particular quadratic we get an **equivalent quadratic in expanded form**, $y = acx^2 + adx + bcx + bd$.
(The familiar "long" multiplication algorithm is an example of the use of the distributive property:
13 x 25 = (3 + 10)(5 + 20) = (3)(5) + 3(20) + 10(5) + (10)(20) = 325.)

	x	5
x	x^2	$5x$
2	$2x$	10

Therefore, $y = (x + 2)(x + 5)$ is **equivalent** to
$y = x^2 + 5x + 2x + 10$, or $y = x^2 + 7x + 10$.

3. Use the distributive property to multiply
(a + b + c)(d + e + f)

	a	b	c
d	ad	bd	cd
e	ae	be	ce
f	af	bf	cf

Therefore, (a + b + c)(d + e + f)
= ad + ae + af + bd + be + bf + cd + ce + cf.

4. Which of the following factored forms are ***quadratic****?*

a) $y = (2x + 3)(x^2 + 5)$
b) $y = (2x + 3)(x – 5)$
c) $y = (2x + 3)(5x – 3)$
d) $y = (2x)(x + 2)$
e) $y = (2)(3 - x^2)$
f) $y = (2x)(x^2 + 3)$

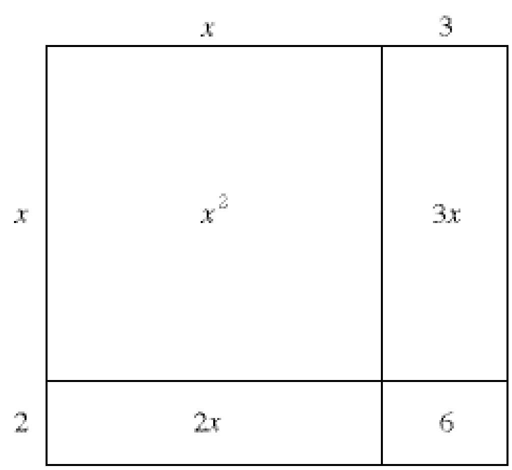
a) If the distributive property is applied to this factored expression (or an area model is drawn) then one of the terms in the resulting expanded form is $2x^3$. So this is not a quadratic relationship.

b) If the distributive property is applied to this factored expression then the term with the highest power in the resulting expanded form is $2x^2$. This is a quadratic relationship.
c) Highest power is in $10x^2$. This is a quadratic relationship.
d) Highest power is in $2x^2$. This is a quadratic relationship.
e) Highest power is in $2x^2$ (actually $-2x^2$). This is a quadratic relationship.
g) Highest power is in $2x^3$. This is not a quadratic relationship.

5. *Write the* ***expanded form*** *$y = 2x^2 + 10x$ in* ***factored form****.*

Y = $2x^2 + 10x$ can be written as $y = 2x(x + 5)$. We say that "2x" is a common factor in both terms of the expanded expression. When we "factor out" the "2x" then the other factor is "x + 5."

6. *Write the* ***expanded form*** *$y = x^2 + 5x + 6$ in* ***factored form****.*



Therefore, $y = x^2 + 5x + 6$ is equivalent to
$y = (x + 2)(x + 3)$.

Note: More time is spent on multiplying binomials and factoring quadratic expressions in the *Say It With Symbols* unit.

Relating patterns of change in a table for a quadratic function to the equation:
In *linear* relationships, the y-value changes at a constant rate, as the x value changes by 1 unit. We say that, if the x-values are changing by a constant increment, then the **first differences** in y are constant, indicating a constant rate of change. In *quadratic* relationships, first differences are not constant, but **second differences**—the differences between successive first differences—are.

Symmetry in the table:
In a table of the coordinate pairs that fit *quadratic* relationships, pairs with identical values of y will be arranged **symmetrically around some "center"** coordinate pair. Sometimes this symmetry is not visible in the table because the "center" pair is not given in the table, perhaps because the increments in the table need to be adjusted, or perhaps because the "center" pair has irrational coordinate values.

Note: this symmetry is a result of the multiplicative nature of quadratic relationships. The same y-value will occur for two different x-values substituted into the equation y = (ax +b)(cx + d) in the same way that y = (3)(5) gives the same result as y = (5)(3).

7. *The perimeter of a rectangle is 20. Make a table showing possible dimensions. What **patterns** appear in the table? What are the dimensions and area of the rectangle with the greatest area?*

If the perimeter P = 20 meters, then the area, A, of the rectangle can be represented as A = LW, where L is the length and W is the width. Since 2(L + W) = 20 or L + W = 10, the area can be written in terms of one of its dimensions, A = L(10 – L). By substituting values for L we can find corresponding values for A. See table below.

L	A	1st diff.	2nd diff.
1	1(10 – 1) = 1(9) = 9		
		16 – 9 = 7	
2	2(10 – 2) = 2(8) = 16		5 – 7 = -2
		21 – 16 = 5	
3	3(10 – 3) = 3(7) = 21		3 – 5 = -2
		24 – 21 = 3	
4	4(10 – 4) = 4(6) = 24		1 – 3 = -2
		25 – 24 = 1	
5	5(10 – 5) = 5(5) = 25		-1 – 1 = -2
		24 – 25 = -1	
6	6(10 – 6) = 6(4) = 24		-3 – (-1)= -2
		21 – 24 = -3	
7	7(10 – 7) = 7(3) = 21		
etc		Not constant	Constant

The Area increases to a **maximum** of 25 and then decreases again. Pairs with identical values for the Area are **symmetrically arranged around the maximum (5, 25)**. The symmetry occurs because at L = 5 both length and width are 5; on either side of the pair (5, 25) in the table are pairs representing rectangles with dimensions 4 by 6 and 6 by 4, which are, of course, congruent rectangles and equal in area. In terms of the equation the coordinate pair at the "center" of the table occurs when the **factors** "L" and "10 – L" are equal, at L = 5. On either side of this "center" pair the **values of the factors are interchanged**; at L = 6, 10 – L = 4, while at L = 4, 10 – L = 6.

The **first differences** are not constant, but the **2nd differences** are all –2. (There is a pattern in the 1st differences, 7, 5, 3 etc, indicating that the Area values increase at a decreasing rate and then start decreasing again.)

8. *What **patterns** appear in the table for* $y = x^2 - 5x + 6$?

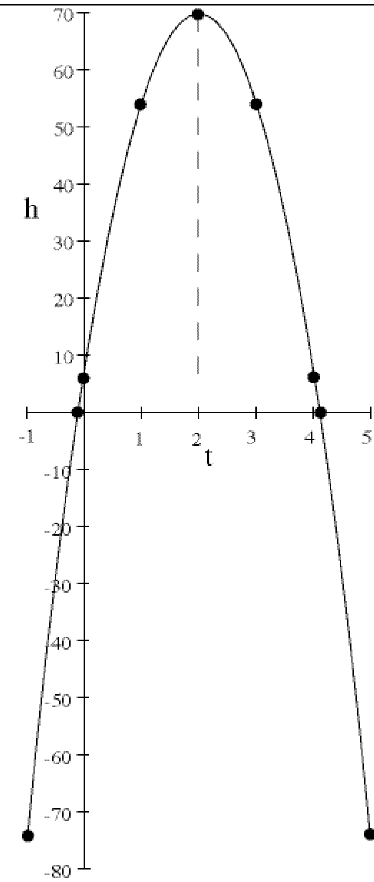
The **factored form** of the equation is $y = (x – 2)(x – 3)$.
It is not necessary to write the equation in this form to see the symmetry in the table. But writing it in this form makes it clear where the symmetry comes from. See bolded part of table.

x	y	1st diff	2nd diff
0	6		
		2 – 6 = -4	
1	**(1 – 2)(1 – 3) = -1(-2) = 2**		-2 - -4 = 2
		0 – 2 = -2	
2	0		0 - -2 = 2
		0 – 0 = 0	
3	0		2 – 0 = 2
		2 – 0 = 2	
4	**(4 – 2)(4 – 3) = (2)(1) = 2**		4 – 2 = 2
		6 – 2 = 4	
5	6		
		Not constant	Constant

The pairs with **identical values for y are symmetrically arranged**. The "center" pair appears to be between (2, 0) and (3, 0). If we had chosen different increments for x we might see exactly where this "center" is. It occurs at (2.5,y). To find the value of y when x = 2.5 we can substitute in the original equation and find y = -0.25.

The **first differences** are not constant. The

	second differences are all 2. Note: This "method of differences" foreshadows the calculus idea of finding the derivative. In general, if $y = ax^2 + bx + c$, then the second differences (setting x-increments at 1 unit) will all be *2a*. In calculus class students will learn to find first derivatives and second derivatives, and to relate these to the rate of change of y, and to the rate of change of the rate of change.
Patterns of change in a graph for quadratic relationships: In *linear* relationships the graph is a straight line because of the constant rate of change in y. For *quadratic* relationships the graph is a curve, indicating the non-constant rate of change in y; the curve will have an **axis of symmetry**; and the **vertex**, indicating the **maximum or minimum** value of y, will lie on this axis. If there are **x-intercepts** then the axis of symmetry will lie halfway between these x-intercepts. (This kind of curve is called a **parabola**.)	*9. The equation $h = {}^{-}16(t)^2 + 64(t) + 6$ models one-dimensional projectile motion. (The equation assumes that the motion occurs in a vacuum, but the predictions will be reasonably accurate for normal conditions.) What* ***patterns of change*** *occur in the graph and what do these patterns tell you about how the height of the projectile is changing over time?* A table of values shows the expected symmetry. t: -1, 0, 1, 2, 3, 4, 5 h: -74, 6, 54, 70, 54, 6, -74

	 The graph representing this relationship has reflection symmetry around a vertical line that passes through (2, 70). This line is the **axis of symmetry.** The point (2, 70) is the **vertex**. In this case, 70 is the maximum value that *h* can reach, so (2, 70) is the **maximum**. The graph illustrates that the height of the projectile increases to a maximum of 70 units and then decreases again. The speed and the time taken for the projectile's motion upward is the same as the speed and the time taken for the descent. It takes 2 seconds for the projectile to travel from a starting height of 6 units to the maximum height. The **x-intercepts** would be the points (t, 0), which do not appear in the table. On the graph we can approximate them as (-0.1, 0) and (4.1, 0). These tell us when the height of the projectile is 0 units, that is, when the projectile is on the ground. This happens after approximately 4.1 seconds. (The other x-intercept does not tell us anything meaningful in this context.)

Summary of information from each format of quadratic equation:

Factored	Expanded
x- intercepts	**y-intercept**
Axis of symmetry	**Steepness of curve**
Maximum/ minimum	**Opens up or down**

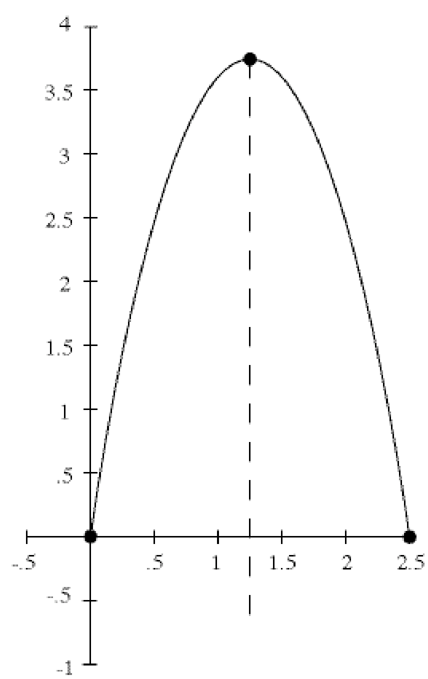
10. What information about the graph of the $y = x(5 - 2x)$ can you deduce from the equation?

The equation is in **factored form**. We can deduce the **x-intercepts** by observing that we can make y = 0 by substituting 0 for x and also 2.5 for x. So there are two x-intercepts: (0, 0) and (2.5, 0).

The **axis of symmetry** must be midway between these points, at x = 1.25.

The **vertex** will be at (1.25, y), on the graph of the parabola. We can find the y-value at the vertex by substituting 1.25 for x into the equation. Therefore, $y = 1.25(5 - 2.5) = 3.75$. The vertex is (1.25, 3.75). It remains to decide if this is a **maximum or a minimum**. If we look at the x-intercepts we see that this point has a higher y-value than either of these, so it can not be a minimum point. (1.25, 3.75) must be a **maximum.**

Putting all this together we have:



11. What information about the graph of y = (x –

	2)(x + 4) can we deduce from the equation? The equation is in factored form. As above we can deduce the **x-intercepts** by observing that we can make y = 0 by substituting x = 2 and x = -4 into this equation. The x-intercepts are (2, 0) and (-4, 0). The **axis of symmetry** is, therefore, x = -1. The **vertex** is at (-1, y), and we can find this y value by substituting x = -1 into the equation, giving y = (-1 – 2)(-1 + 4) = (-3)(3) = -9. The vertex is (-1, -9). To decide if this is a **maximum or a minimum** we could compare it to the x-intercepts; alternatively we could find points close to the vertex, on either side of the vertex, to get a picture of what is happening at the vertex. Thus, we might try x = -2 and x = 0, producing points (-2, -8) and (0, -8). These 2 points are **symmetrically arranged around and above the vertex** and so this time we have a **minimum**. *12. The **expanded forms** of the quadratic equations in examples 10 and 11 are:* *a) $y = 5x – 2x^2$ and* b) *$y = x^2 + 2x – 8$.* *What new information can be deduced from this expanded form?* a) From the expanded form we see that, when x = 0, y = 0. Therefore, the **y-intercept** is (0, 0). We can also deduce that the graph will **open downwards** because the **x-squared term, -2*x*^{2}, has a negative coefficient.** (Students make this observation from many examples in class and homework.) From the down-facing orientation we deduce that the graph has a **maximum**, though the exact position of the maximum is easier to deduce from the factored form. b) From the expanded form we see that, when

	x = 0, y = -8. Therefore the **y-intercept** is (0, -8). From the term $1x^2$ we deduce that the curve will **open upwards,** and so the **vertex will be a minimum**, though, again, the exact location of the vertex will be easier to find from the factored form. We can also deduce that $y = 5x – 2x^2$ will be a steeper and narrower parabola than $y = 1x^2 + 2x – 8$. This information comes from the coefficient *a* of the ax^2 term. As $\|a\|$ increases the curve becomes narrower and steeper. (Again, students observe this from the many examples they see in class and homework.) Note: students have many more opportunities to become comfortable with factoring quadratics and solving quadratic equations in *Say it With Symbols*.