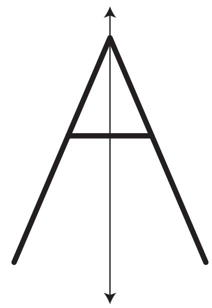
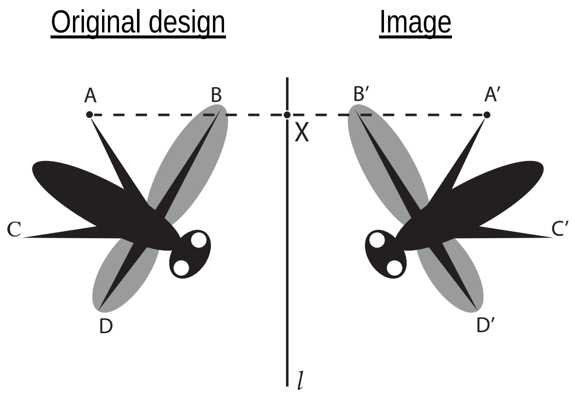
Vocabulary: *Hubcaps, Kaleidoscopes and Mirrors*

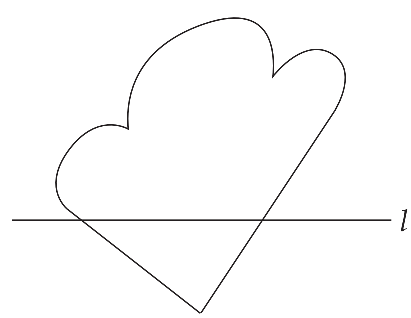
Concept	Example
Two related ideas: Symmetry and Transformation. **Symmetry** is a *property* of some designs or shapes. A design either has symmetry or does not. For example, the letter A has symmetry, while the letter P does not. Three kinds of symmetry are investigated in this unit: **reflection**, **rotation**, **translation.** A **transformation** is an *action* that acts upon a design (whether it has symmetry or not) and creates an image of that design. Sometimes the original design + the image created by a transformation make a new design with symmetry. Reflection Symmetry A design has **reflection symmetry**, also called **mirror symmetry**, if a reflection in a line matches the figure exactly onto itself. The line is called the **line of symmetry** (or axis of symmetry or line of reflection)). Each point on one part of the design matches a point on the other side of the line of symmetry. We say the matching points are **images** of each other. Reflection Transformation A reflection transformation can be specified by giving the **line of reflection.** Under a reflection in a line *l* the point *A* and its **image point** *A′* ✓ lie on a line that is perpendicular to the line of symmetry and ✓ are equidistant from the line of symmetry. Note: Any design, whether symmetric or not, can be reflected in a chosen line, with the result that the new design created (the original + the image) will have reflection symmetry.	1. The letter A below has **reflection symmetry** because a reflection in the vertical line will match each point on the left half with a point on the right half. The vertical line is the **line of symmetry** for this design.  <u>Tools to Check for Reflection Symmetry</u> • Transparent reflection tools, such as Image Reflectors, allow the viewer to see a reflected **image** of one part of a design while simultaneously looking at the entire original design. When the image of the part of the design in front of the plastic matches the part of the design behind the plastic, the **line of symmetry** can be identified by drawing a line segment along the bottom edge of the plastic. • Mirrors allow the viewer to see one part of a design with its the reflected **image**. However, the viewer has to rely on visual judgment to decide if the new design, made of the visible part of the design together with its reflected image, is identical to the original design. • Tracing paper can be used to trace the entire design. If the tracing can then “flipped” to fit exactly on the original then the design has reflection symmetry. However, the line of symmetry is not obvious with this tool.

2. The original design on the left does not have reflection symmetry. It has been **reflected** in the vertical line. The image created by using this **transformation** is shown on the right. The original design, taken together with its image on the right, create a new design with **reflection symmetry.**

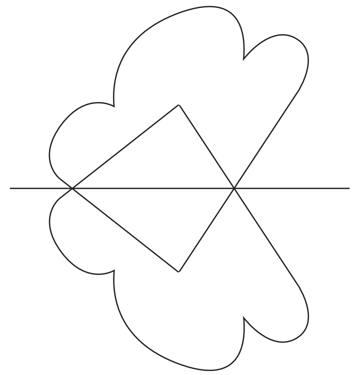


A' is the image of A, B' is the image of B etc.
The distance AX and A'X are the same. AA' is perpendicular to the line of symmetry. Since AA' is bisected by the line of symmetry, and since AA' is perpendicular to the line of symmetry, we say that the line of symmetry is the **perpendicular bisector** of AA'.

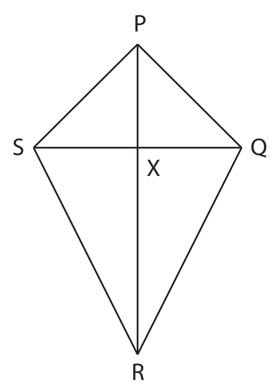
3. *Create a design with **reflection symmetry** by using the shape below and the **given line of symmetry.***



The result is a design with reflection symmetry n the line l.



4. *This kite has* ***reflection symmetry****. What is the* ***line of symmetry****? What is the image of point Q under this reflection? Point P? What distances can we conclude are equal because of the reflection symmetry?*



The line of symmetry is PR. The image of point Q is point S. The image of point P is point P. We can conclude that distances from points to the line of symmetry match distances from corresponding images to the line of symmetry. So QX = SX. We can also conclude that PS = PQ because PS is the image of PQ. Likewise, QR = SR.

Rotation Symmetry

A design has **rotation symmetry** if a rotation other than a full turn about a point matches the figure onto itself. The point around which the design rotates is called the **center of rotation**, and the least amount of turn necessary to produce a match is called the **angle of rotation.**

Rotation Transformation

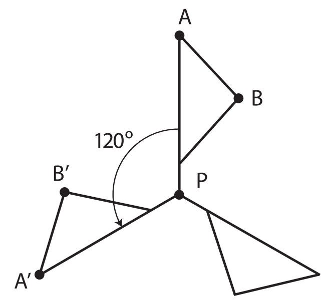
A rotation transformation can be specified by giving the **center of rotation** and the **angle of rotation**. Under a rotation around a point P, the point A and its **image** A'

- ✓ will be the same distance from the center of rotation, P, and
- ✓ the angle of rotation is APA' and is the same for all points under this rotation (that is, angle APA' is the same as BPB' etc.).

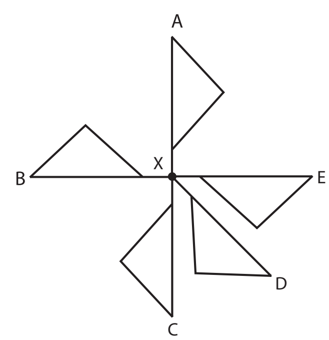
Note: Any design can rotated around a center of rotation, by some chosen angle of rotation, to create an image. However, the resulting design does not necessarily have rotation symmetry.

5.

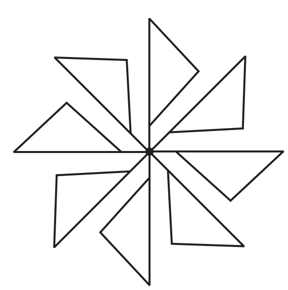
The design below has **rotation symmetry** because a rotation of 120° or 240° about point P will match each flag with another flag. Point P is referred to as the **center of rotation**. The **angle of rotation** for this design is 120°, the smallest angle through which the design can be rotated to match with the original design.

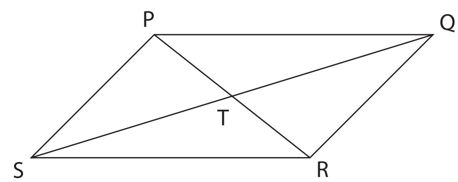


6. *The figure below does not have* ***rotation symmetry****. Can you complete the design so that it will have rotation symmetry? What is the* ***center of rotation****? What is the* ***angle of rotation****?*



One answer is as shown below. The angle of rotation is 45 degrees, and the center of rotation is X.



	7. *This parallelogram has* ***rotation symmetry****. What is the center of rotation? What is the angle of rotation? What point is the* ***imag****e of R? What distances must be equal because of the rotation symmetry?*  The center of rotation is T. The angle of rotation is 180 degrees. P is the image of R. PT must be the same distance as RT since P is the image of R and T is the center of rotation. Likewise, QT = ST. (You can also deduce that PS = RQ and PQ = RS)

Translation Symmetry

A design has **translation symmetry** if a translation, or slide, matches the figure onto itself.   Note: in order for this to be true the design would have to continue indefinitely.

Translation Transformation

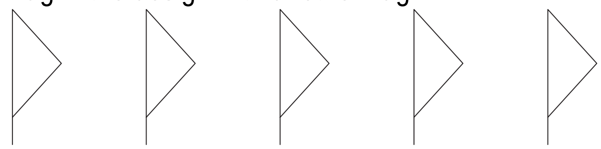
A translation transformation can be specified by giving the **direction** and the **distance** of the translation or *slide*. The result of a translation in the direction and distance of a given line segment *l*, is that the point A and its **image** A'

- ✓ are the same distance apart as the length of segment *l,* and
- ✓ AA' is parallel to the direction of the translation, and parallel to BB' etc.

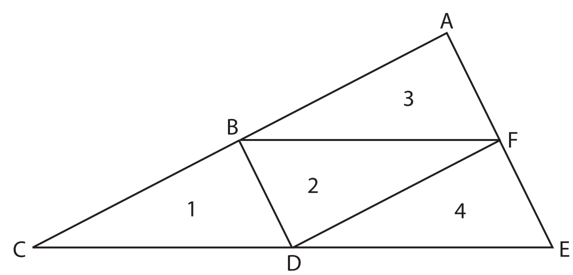
Note: Any finite design can translated a given direction and distance to create an image. However, the resulting design does not have translation symmetry.

8.

The figure below is part of a translation-symmetric design. If this design continued in both directions, a slide of 1 unit to the right or left would match each flag in the design with another flag.



9. *The large triangle below is made of triangle 1 and 3 copies of triangle 1. Using translation alone to transform triangle 1, can the image of triangle 1 match triangle 2? Can the image of triangle 1 match triangle 3? Triangle 4? If the answer is "yes" then describe the translation.*



Under translation *alone*, triangle 1 can be matched to triangle 3 or triangle 4, but not triangle 2, because triangle 2 is oriented differently (points downwards). To translate triangle 1 so that the image matches triangle 3 we need to use side BC as a guide for the distance and direction: C slides to B, B slides to A etc. To translate triangle 1 so that the image matches triangle 4 we need to use CD as a guide for the distance and direction: C slides to D, D slides to E etc.

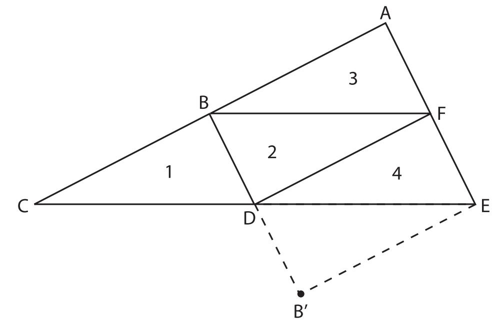
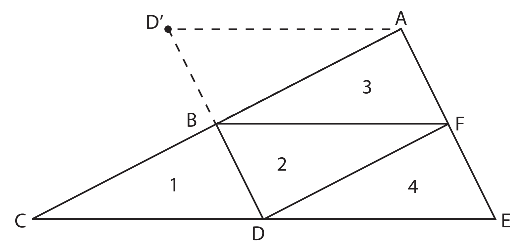
Combinations of transformations

The transformations described above can be used in combinations, each image being created from the last image, in a succession of actions. The *order* of these transformations is important. (See "coordinate representations" below.)

10. *What single transformation* ***or combination of transformation****s, acting on triangle 1(see example 9), will create an image that matches triangle 2?*

We could

- rotate triangle 1 180 degrees around point D and then translate the image along B'D to match triangle 2.

	 or • rotate triangle 1 180 degrees around the midpoint of BD, or • rotate triangle 1 180 degrees around point B and then translate along D'B. 

Congruence of Shapes

If two shapes can be matched using reflections or rotations or translations then the shapes are congruent, that is the shapes are identical in every respect except, possibly, orientation.

These transformations are called *isometric* or shape- and size- preserving.

Congruent Triangles

- If two triangles are congruent then 3 pairs of corresponding angles have the same measures, and 3 pairs of corresponding sides have the same lengths.
- To ensure that 2 triangles are congruent we need only to match a subset of the corresponding angles and sides. If we know that
    1. 3 sides of 1 triangle match 3 sides of another triangle then the triangles are congruent. (SSS)
    2. 2 sides and the included angle of one triangle match 2 sides and the included angle of another triangle then the triangles are congruent. (SAS)
    3. 2 angles and a side of one triangle match 2 angles and a corresponding side of another triangle then the triangles are congruent. (AAS or ASA)

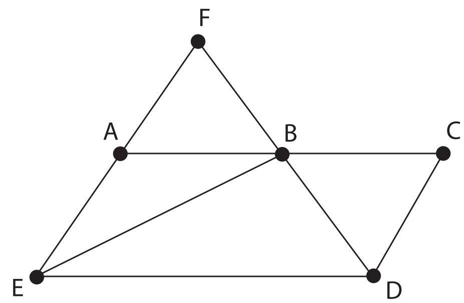
Note: If we know that all three angles of one triangle match the three angles of another triangle then the triangles will be similar, that is the same shape, but not necessarily congruent; they will be scale copies of each other, the scale to be determined by comparing corresponding sides. (See *Stretching and Shrinking*.)

Note: To investigate whether two quadrilaterals are congruent we can divide

11. *Is triangle 1 congruent to triangle 2 in the sketch in example 9?*

Yes. If we can match a copy of a triangle 1 to triangle 2, using reflections or rotations or translations, then the triangle 1 and triangle 2 are congruent.

12. *In the figure below we know only that in triangle EFD, A is the midpoint of side EF, and B is the midpoint of side FD, and we know that ACDE is a parallelogram. Use a congruence theorem about triangles to prove that a pair of triangles are congruent.*



There are two congruent triangles in this figure; they are triangle AFB and triangle CDB.

- ✓ Angle ABF = angle CBD, because these are vertically opposite angles
- ✓ Angle AFB = angle CDB, because DC is parallel to EA or AF (opposite sides of a parallelogram)
- ✓ FB = DB, because B is a midpoint of FD

Thus 2 angles and a side of one triangle match corresponding angles and side of the other triangle. The triangles are congruent.

13. *Suppose in the figure above we know only that in triangle EFD, A is a midpoint of side EF and B is a midpoint of side FD, and that B is a mid point of AC. Could we still show that triangles AFB and CDB are congruent?*

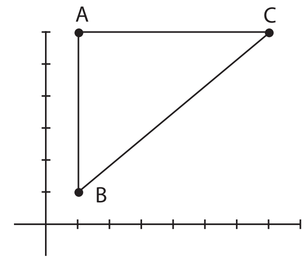
Students could argue by using a side-angle-side combination. OR they could use a transformation

the quadrilaterals into triangles and compare the triangular parts.	argument, as follows: ✓ F is the image of D under a 180 degree transformation around B. (We know that FB = DB) ✓ Likewise A is the image of C under 180 degree transformation around B. ✓ And B is the image of B. ✓ So triangle AFB is the image of triangle CDB under a 180 degree rotation around B. That is, the triangles are congruent.

Coordinate Rules for Transformations

When the point (x,y) is transformed …	The resulting image is …
Under a reflection in the x axis	(x, -y)
Under a reflection in the y-axis	(-x, y)
Under a reflection in the line y = x	(y, x)
Under a counterclockwise rotation of 90 degrees around (0, 0)	(-y, x)
Under a rotation of 180 degrees around (0,0)	(-x, -y)
Under a translation of c units in the direction of the x-axis	(x + c, y)
Under a translation of d units in the direction of the y-axis	(x, y + d)
Under a translation in the direction of the line y = x)	(x + s, y + s) that is, a horizontal and a vertical translation of the same distance.

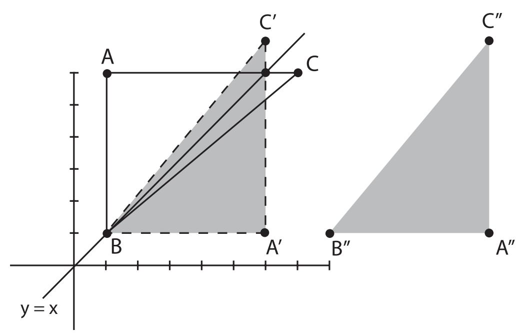
14. *Apply the coordinate rules for transformations to find the image of triangle ABC under a reflection in the line y = x, and then a translation, of the first image, 7 units in the direction of the x axis. What is the point A'' (second image)? B''? C''?*



Using coordinate rules to track transformations we have:

Point	Reflection in y = x	1st image	Translation 7 units right	2nd image
A(1,6)	→	A'(6,1)	→	A"(13,1)
B(1,1)	→	B'(1,1)	→	B"(8,1)
C(7,6)	→	C'(6,7)	→	C"(13,7)

We can follow the transformations visually, as below:



Note: When two transformations are used consecutively, as in example 14, the order may impact the result. If the figure in example 14 were translated 7 units right first, and then reflected over y = x, we would have a different result.
(1, 6) → (8, 6) → (6, 8). Sometimes it is easier to track a combination of transformations by using coordinates that by making drawings.