Filling and Wrapping: Homework Examples from ACE

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Investigation 1: *Building Smart Boxes: Rectangular Prisms*
ACE #3

Suppose you plan to make a box that will hold exactly 40 one-inch cubes.

a. Give the dimensions of all the possible boxes you can make.
b. Which of the boxes described in part (a) has the least surface area? Explain.

For this problem, students can visualize a box and decide how many cubes are on the base layer, and how many layers high the box is. Since we are talking about whole unit cubes we must look for factors of 40; one factor is the base layer and the other factor is the height. That is: *base layer* x *height* = 40. Also, *length* x *width = area of base layer*.

There may be more than one choice for length and width for a given base layer. For example, we know that 5 x 8 = 40, so the base layer could have 5 cubes and the height could be 8 inches. If the base layer has 5 inch cubes then the length must be 5 inches and the width must be 1 inch (or vice versa). Or, since we also know that 10 x 4 = 40, the base layer could be 10 inch cubes and the height could be 4 inches. If the base layer has 10 inch cubes then the length might be 10 inches and the width 1 inch, or the length might be 5 inches and the width 2 inches. We can organize the dimensions of all the possible boxes in a table (L is length, W is width, H is height, V is volume, and SA is surface area, which we can calculate using the dimensions).

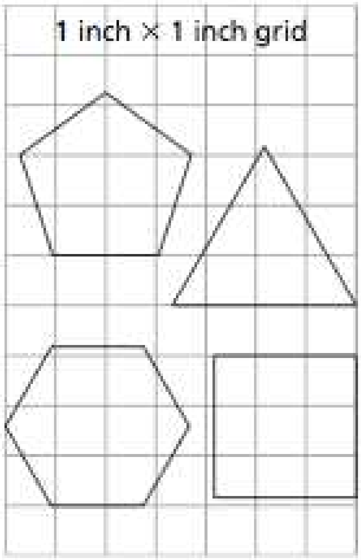
Base	L	W	H	V	SA
1	1	1	40	40	162
2	1	2	20	40	124
4	1	4	10	40	108
4	2	2	10	40	88
5	1	5	8	40	106
8	1	8	5	40	106
8	2	4	5	40	76
10	1	10	4	40	108
10	2	5	4	40	76
etc.					

If we examine this table we will see that some of the boxes created are identical. *L* = 1, *W* = 4, *H* = 10 will be exactly the same box as *L* = 1, *W* = 10 and *H* = 4. There are only 6 unique arrangements.

b. The 2 x 4 x 5 box has the least surface area. We can tell this is true since we already calculated the surface areas of all the possible boxes. But it's also the box closest to a cube, which would have the smallest surface area.

Investigation 2: *Polygonal Prisms*
ACE #12

For Problem 2.1, Sheryl made paper prisms that were all 8.5 inches high. She traced the polygon bases on 1-inch grid paper to give a picture like the one shown below. Estimate the volume of each prism.



First, a note: We know that each of the bases must have the same perimeter, because they were all made by folding the same piece of paper, which was 11 inches long. But that doesn't mean their areas are the same. This refers back to an idea from *Covering and Surrounding*—equal perimeters do not imply equal areas.

For every prism, the formula is ***volume = base area x height.*** We know from the given information that the height is 8.5 inches for each of these prisms, which makes sense because the problem instructed student to use an 8.5 x 11 inch piece of paper for each prism. This problem asks for estimates of the volumes, which requires students to estimate the base areas using the grid. Students can use a variety of strategies to estimate the areas of the bases. Here, we explain possible volumes using formulas for area, though students do not need to use these calculations.

- The square has sides of $\frac{11}{4}$ or 2.75 inches. Therefore, the *base area* = 2.75 · 2.75 = 7.5625 square inches, and *volume* = 7.5625 · 8.5 = 64.28125 cubic inches.

- Each side of the triangle is $\frac{11}{3}$ or about 3.67 inches long. The height appears to be about 3.2 inches (students need to estimate this value, because they don't yet know how to use the Pythagorean Theorem.) Therefore, *the base area* = $(\frac{1}{2})\ (\frac{11}{3})\ (3.2) \approx$ 5.87 square inches and *volume* ≈ 5.87 · 8.5 = 49.9 cubic inches.

- The pentagonal base is made of a 2 x 2 square and 3 triangles surrounding the square.

The areas of the triangles can be approximated using a method from *Covering and Surrounding*, estimating the height and/or the base of the triangle. The sum of the areas of these triangles ≈ (0.5)(3.5)(1.25) + (0.5)(2)(0.75) + (0.5)(2)(0.75) ≈ 3.6875 square inches. So the *base area* ≈ 7.6875 square units and volume ≈ 65.34 cubic inches.

- You can visualize this hexagon as being made of 6 small equilateral triangles with side lengths of $\frac{11}{6}$. (See *Shapes and Designs* for more explanation about this.) Again, students will have to approximate the height of each of these as about 1.6 units, which we can estimate because the whole hexagon is a little over three inches tall. So the area of each small triangle is $(0.5)(\frac{11}{6})(1.6) \approx 1.47$ square units. Therefore, the total base area ≈ 6(1.47) = 8.8 square units and the volume ≈ (8.8)(8.5) = 74.8 cubic units approx.

Note: As the same 11 inches (on a side of paper) is folded to make more and more sides of a polygon, the base area increases and, therefore, the volume increases.

Investigation 3: *Area and Circumference of Circles*
ACE #11

A pizzeria sells three different sizes of pizza. The small size has a radius of 4 inches, the medium size has a radius of 5 inches, and the large size has a radius of 6 inches.

a. Copy and complete the table below. Explain how you found the areas of the pizzas.
b. Jamar claims the area of a pizza is about 0.75 x (diameter)^2. Is he correct? Explain.

Pizza Measurements

Pizza Size	Diameter (in.)	Radius (in.)	Circumference (in.)	Area (in.^{2})
Small	■	■	■	■
Medium	■	■	■	■
Large	■	■	■	■

a. Using the formula for area of a circle *area* = 3.14r^2 (here, π is approximated to be 3.14) and the formula for circumference of a circle *circumference* = πd, we get the following values:

Pizza Size	Diameter (in.)	Radius (in.)	Circumference (in.)	Area (in.^{2})
Small	8	4	3.14(8) = 25.12	3.14(4^2) = 50.24
Medium	10	5	3.14(10) = 31.4	3.14(5^2) = 78.5
Large	12	6	3.14(12) = 37.68	3.14(6^2) = 113.04

b. Jamar's estimate is close, but will be a slight underestimate, since π is a little more than 3. But $A = \pi r^2 = \pi(\frac{d}{2})^2 \approx \frac{3d^2}{4}$ = 0.75d^2. Students don't need to offer this sort of algebraic reasoning. It is okay to check a few cases to see if the numbers turn out the same or close.

Investigation 3: *Area and Circumference of Circles*
ACE #22

A rectangular lawn has a perimeter of 36 meters and a circular exercise run has a circumference of 36 meters. Which shape will give Rico's dog more area to run? Explain.

This question refers back to the idea in earlier investigations that two shapes with the same perimeter do not necessarily have the same area.

Students discovered that when you compare rectangles, the more "square" a rectangle is (basically, the closer the ratio of sides is to 1:1) the more area it can enclose for a given perimeter. So in this case the "best" rectangle they can make is a 9 meter x 9 meter square. The perimeter of the square is 9 + 9 + 9 + 9 = 36 meters, and the area is 9^2 = 81 square meters.

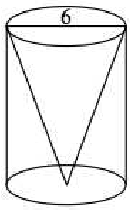
Now we have to do some reasoning with the formula for the circumference of a circle: $C = \pi d$. So $36 = \pi d$, and the diameter is $\frac{36}{\pi} \approx 11.5$ meters. The radius is about $\frac{1}{2}(11.5) = 5.75$ meters. Now that we know the radius, we can figure the area of the circle: *area* = $\pi(5.75)^2 \approx$ 103.9 square meters.

Therefore, a circle with a circumference of 36 meters covers more area than a square with perimeter 36 meters!

Investigation 4: *Cylinders, Cones, and Spheres*
ACE #13

The track and field club is planning a frozen yogurt sale to raise money. They need to buy containers to hold the yogurt. The two options are the cup and cone shown below. The two containers have the same cost. The club plans to charge $1.25 for a serving of yogurt. Which container should the club buy? Explain.

Students have a formula for the volume of a cone; they found that a cone was $\frac{1}{3}$ the volume of a cylinder with the same base and height.



The volume of the cylinder is: V = base area x height = $\pi(3^2) \times 12 \approx 339.3$ cubic centimeters. So, the volume of the cone is $\frac{339.3}{3} = 113.1$ cubic centimeters.

The volume of the cylindrical cup is $\pi(2.5^2)(4.5) = 88.36$ cubic centimeters.

So, the cone is the better buy from the customers' point of view—you get more yogurt for the same $1.25. But, from the club's point of view, the cup holds less, so it will generate more profit—customers pay the same but get less yogurt. The club should choose the cylindrical cup.

Investigation 4: *Cylinders, Cones, and Spheres*
ACE #28

Chilly's Ice Cream Parlor is known for its root beer floats.

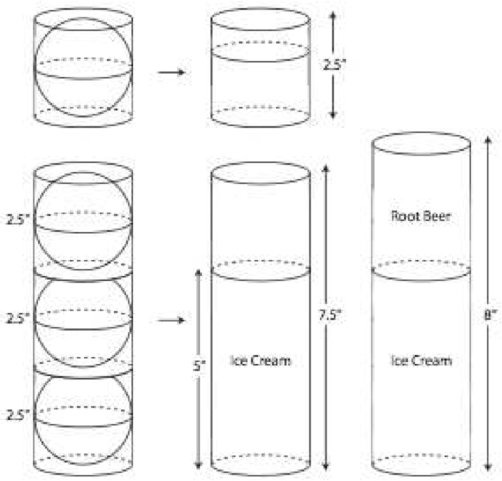
- The float is made by pouring root beer over 3 scoops of ice cream until the glass is filled $\frac{1}{2}$ inch from the top.
- A glass is in the shape of a cylinder with a radius of $1\frac{1}{4}$ inches and height of $8\frac{1}{2}$ inches.
- Each scoop of ice cream is a sphere with a radius of $1\frac{1}{4}$ inches.

Will there be more ice cream or more root beer in the float? Explain.

There are at least two different ways to think about this problem. One is to calculate the volume for the cylinder-shaped float and subtract the volume of the three spheres of ice cream using volume formulas. The other is to visualize and use the relationships between the cylinder and spheres. Each method is shown below.

1) Calculating volumes using formulas:
   - The first 2 clues tell us that the total volume of the float is the volume of a cylinder with height 8 inches (since it's filled to $\frac{1}{2}$ inch from the top) and radius 1.25 inches. To find the volume of this cylinder: $V = \pi(1.25^2)(8) \approx 39.27$ cubic inches.
   - The third clue tells us that the volume of the ice cream is 3 times the volume of a sphere with radius 1.25 inches. Students should have already found out that the volume of a sphere is $\frac{2}{3}$ the volume of a cylinder with the *same height and radius*. **The height of a sphere is the same as the diameter**, which in this case is 2.5 inches. So each sphere of ice cream has volume $\frac{2}{3}$ times the volume of a cylinder with radius 1.25 and height 2.5. Using the formula for the volume of a cylinder: $V = (\frac{2}{3})(\pi)(1.25^2)(2.5) \approx 8.18$ cubic inches. Three scoops of ice cream have a volume of (3)(8.18) = 24.54 cubic inches.
   - We can find the volume of the root beer by subtracting the volume of the ice cream from the total volume of the float. So the root beer volume is 39.27 – 25.54 = 14.72 cubic inches. Thus, there is more ice cream than root beer in the float.

Alternative method:
If *one* sphere of ice cream has $\frac{2}{3}$ the volume of a cylinder with the same radius and height, then *three* scoops will have the volume of 2 cylinders with same radius and height. Therefore, 3 scoops of ice cream have the same volume as 2 cylinders with radius 1.25 and height 2.5 inches, or the same as *one* cylinder with the same radius and double the height—that is, radius 1.25 and height 5 inches. The float in the problem is a cylinder with radius 1.25 and height 8 inches! If 5 inches of this height is packed with ice cream, then only 3 inches space is left for the root beer (see the visual below). So this is another way to determine there is more ice cream than root beer in our float.


2.5"
2.5"
2.5"
2.5"
5"
Ice Cream
7.5"
Root Beer
Ice Cream
8"