

Thinking With Mathematical Models: Homework Examples from ACE

Investigation 1: *Exploring Data Patterns*, ACE #1

Investigation 2: *Linear Models and Equations*, ACE #4

Investigation 3: *Inverse Variation*, ACE #9

Investigation 4: *Variability and Associations in Numerical Data*, ACE #5

Investigation 5: *Variability and Associations in Categorical Data*, ACE #16

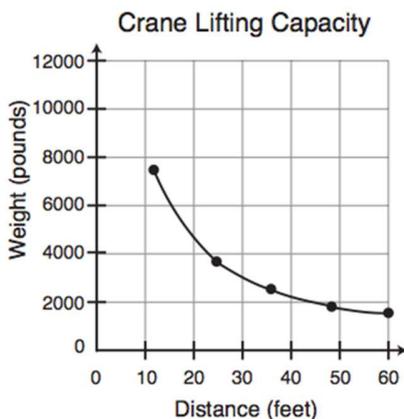
Investigation 1: *Exploring Data Patterns* ACE #1

The table shows the maximum weight a crane arm can lift at various distances from its cab. (See diagram in student text.)

- Describe the relationship between distance and weight for the crane.
- Make a graph of the (distance, weight) data. Explain how the graph's shape illustrates the relationship you described in part a.
- Estimate the weight the crane can lift at distances of 18, 30, and 72 feet from the cab.

a. As the distance from cab to weight increases, the weight decreases. But the rate of change is not constant. For every 12 feet increase in the distance from the cab, the weight decreases, but not by the same amount every time. That is, the *change* in the weight is less and less every time.

b. The graph shows the weight decreasing as distance increases.



The curve of the graph shows that the initial decreases in weight are much larger than later decreases in weight.

c. 18 feet is half way between 12 and 24 feet. So the predicted weight for 18 feet should be between 7500 and 3750 pounds. If the weight decreased at a constant rate, the predicted weight would be 5625 pounds, exactly half way between 3750 and 7500 pounds. BUT the weight seems to be falling at a faster rate at the start, so the correct prediction is probably closer to 3750 than to 7500 pounds. At this point any prediction between 5625 and 3750 pounds would be sensible. (Later students will know more about this pattern, and be able to make better predictions.)

The same reasoning as above would put the predicted weight between 3750 and 2500 pounds, but closer to 2500 pounds. This time we don't have collected data on either side of 72 feet. The predicted weight has to be less than 1500 pounds. Students might note that the weight decreased by 375 pounds from distance 48 to distance 60 feet. 72 feet is 12 feet more than 60 feet, so some might predict that the weight would be $1500 - 375 = 1125$ pounds. Since the weight is decreasing, but at a decreasing rate, this prediction is too low. The weight held by the bridge decreased as the length increased, but not at a constant rate.

Note: Students learn that this kind of relationship is called an *inverse proportional relationship*. This means that as the independent variable increases the dependent variable decreases, but not at a constant rate; in fact the product of the independent and dependent variables is a constant, for example, $xy = 10$ or, in general, $xy = a$ (which can also be written as $y = ax$ etc.).

Investigation 2: Linear Models and Equations

ACE #4

The table gives average weights of purebred Chihuahuas from birth to age 16 weeks (See student text).

- Graph the (age, weight) data, and draw a line that models the data pattern.
- Write an equation in the form $y=mx+b$ for your line. Explain what the values of m and b tell you about this situation.
- Use your equation to estimate the average weight of Chihuahuas for odd-numbered ages from 1 to 15 weeks.
- What average weight does your linear model predict for a Chihuahua that is 72 weeks old? Explain why this prediction is unlikely to be accurate.

a. When trying to decide where to draw a line that fits the data pattern, one wants to not let any one point be too influential. All points should “pull” on the line, so that the placement of the line reflects an overall trend. There should be about the same number of data points above as below the line. One tries to adjust the placement of the line so that the “gaps” between the line and the data points are minimized. The line may pass through several data points, or just a few points, or “miss” all points.

b. The equation given here should fit whatever line is drawn in part a. From hand-drawn lines we can figure slope by reading two points that seem to be exactly on the line. In this case the line might pass through (6, 17.5) and (12, 30) making the slope $= \frac{12.5}{5} = 2.08$. Students might read the intercept from the graph (looks like approximately 5) or use the calculated slope to count back to the y -intercept. The “ $y = mx + b$ ” equation produced should have the *calculated* slope and intercept in place of “ m ” and “ b ” respectively. The intercept tells us the average weight of a Chihuahua at age zero, that is at birth. The slope tells us how much an average Chihuahua is expected to grow each year. Student equations will vary but should be similar to $W = 2.08A + 5$.

c. Substituting age = 1 into the above equation we have $W = 2.08(1) + 5 = 7.08$ ounces. The rest of the table is found by substituting appropriate values for age. Notice that these values will differ if a different line has been drawn in part a. However, the weight values found by students should be similar to those shown.

A	1	3	5	7	9
W	7.08	11.24	15.4	19.56	23.72

d. Substituting $A = 72$ into the above equation, we have $W = 2.08(72) + 5 = 154.76$ ounces or about 9.7 pounds. This is unreasonably heavy, not a good representation of an “average” weight of a Chihuahua that is 72 weeks (over 1 year) old. This illustrates that mathematical models, or in this case a line of best fit, can not be trusted to continue to model the data well when we stray too far from the given data.

Investigation 3: *Inverse Variation*
ACE #9

Testers drove eight vehicles 200 miles on a track at the same speed. The table below shows the amount of fuel each car used.

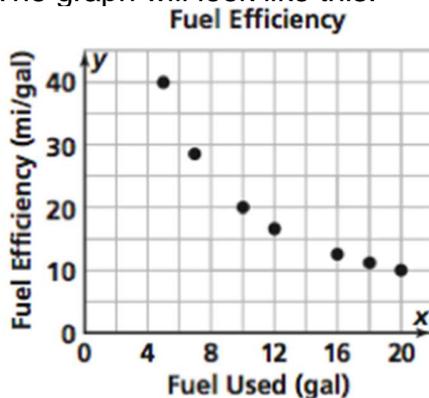
- a. Find the fuel efficiency in miles per gallon for each vehicle.
 - b. Make a graph of the (fuel used, miles per gallon) data. Describe the pattern of change shown in the graph.
 - c. Write a formula for calculating the fuel efficiency based on the fuel used for a 200-mile test drive.
 - d. Use your formula to find how fuel efficiency changes as the number of gallons of fuel increased from 5 to 10, from 10 to 15, and from 15 to 20.
 - e. How do the answers for part (d) show that the relationship between fuel used and fuel efficiency is not linear?
- a. To find the fuel efficiency for each vehicle, students will divide the total distance of the trip (200 miles) by the value in the “Fuel Used” column. For example, the Large Truck used 20 gallons of fuel during the 200 mile trip. $200 \text{ (miles)} \div 20 \text{ (gallons)} = 10 \text{ (miles per gallon)}$

The table will look like this:

Vehicle Type	Fuel Used (gal)	Fuel Efficiency (mi/gal)
Large Truck	20	10
Large SUV	18	11.11
Limousine	16	12.5
Large Sedan	12	16.67
Small Truck	10	20
Sports Car	12	16.67
Compact Car	7	28.57
Sub-Compact Car	5	40

- b. Students can make the horizontal axis represent the amount of fuel used (in gallons), and the vertical axis represent the fuel efficiency that they just calculated in part (a). Each vehicle is then represented by one point on the graph. For example, the Large Truck is represented by the point (20, 10) because it used 20 gallons of gas, and its fuel efficiency in part (a) is 10 miles per gallon.

The graph will look like this:



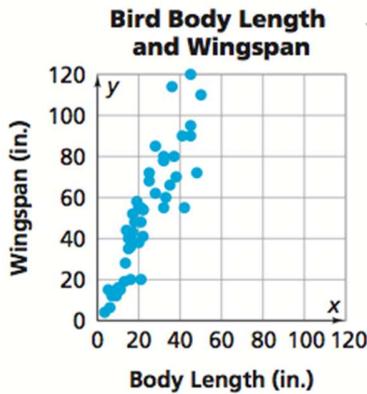
c. To find the formula, first notice that the graph displays inverse variation between fuel used and fuel efficiency. The general formula for an inverse variation is $y = \frac{k}{x}$ or $xy = k$. We can use any point on the graph to find the value of k . For example, the point that represents the Large Truck is (20, 10). Substitute 20 for x , and 10 for y in the equation $y = \frac{k}{x}$ and solve for k . It turns out that $k = 200$. We can then write a formula for fuel efficiency, e , as:
 $e = 200 \div f$, where f is the amount of fuel used

d. To determine how fuel efficiency changes between two values, substitute each value separately into the equation from part (c) and subtract. For example, to see how fuel efficiency changes as the number of gallons of fuel used goes from 5 to 10, first substitute 10 into the equation from part (c). This tells us that $e = 200 \div 5 = 40$ miles per gallon. Next, substitute 5 into the equation. This tells us that $e = 200 \div 10 = 20$ miles per gallon. The difference between these values is $40 - 20 = 20$ miles per gallon. Between 10 and 15 gallons of fuel, fuel efficiency decreases by 3.33 mpg; Between 15 and 20 gallons of fuel, fuel efficiency decreases by 6.67 mpg.

e. If the relationship between the amount of fuel used, and fuel efficiency was linear, we would expect the amount of change between two values to be constant, as long as the difference in the original two values was the same. The results of part (d) tells us that this is not true. The amount of change in fuel efficiency when going from 5 to 10 gallons of fuel used is different than when going from 10 to 15 gallons of fuel used, despite the fact that the difference between 10 and 5 is the same as the difference between 15 and 10. Constant change in fuel used does not lead to constant change in fuel efficiency.

Investigation 4: *Variability and Associations in Numerical Data*
ACE #5

The scatter plot below shows the relationship between body length and wingspan for different birds.



a. Use your results from Exercise 3. Does your equation for the relationship between height and arm span also describe the relationship between body length and wingspan for birds?

b. Find a line that fits the overall pattern of points in the scatter plot. What is the equation of your line?

c. Predict the wingspan of a bird with a body length of 60 inches. Explain your reasoning.

a. In Exercise 3, students find that the equation that relates height and armspan is $armspan = height$. In this scatterplot, the equation $wingspan = length$ does not seem a good fit for the scatter plot data. If the two variables were roughly equal to each other, we would expect the scatter of points to run diagonally from (0,0) to (100,100). Instead, we see that the wingspan measurement (in inches) are roughly twice the corresponding body length.

b. Estimates of a good fit linear model will vary. $W = 2L$ is a pretty good rough estimate. This line has y -intercept (0, 0) and slope 2, meaning that wingspan increases twice as fast as body length.

c. Using the estimated linear model offered in (b), the predicted wingspan would be: $W = 2L = 2(60) = 120$ inches. Therefore, our model predicts that a bird with a body length of 60 inches would have a wingspan of 120 inches.

Investigation 5: *Variability and Associations in Categorical Data*
ACE #16

Use the survival rate data of men, women, and children on the *Titanic*

Passenger Category	Saved	Lost
Men	338	1,352
Women	316	109
Children	56	53

Which of these claims about survival rates on the *Titanic* are true? Explain your reasoning.

a. More men than women were saved.

b. Women were more likely than children to be lost.

c. Men were about six times as likely to be saved as women.

a. While it is true that more men were saved than women (338 versus 316), there were far more men on the boat ($338 + 1352 = 1690$) than women ($316 + 109 = 425$). Therefore the fraction of men saved ($338/1690 = 20\%$) was less than the fraction of women saved ($316/425 = 74\%$)

b. False. About $1/4$ of total women lost ($109/425$); There were 109 children on the boat ($56 + 53$), so about 50% of children were lost ($53/109$). Therefore children were more likely than women to be lost.

c. It is true that about six times as many men were saved. 56 children were saved, and $56 \times 6 = 336$. This number is very close to the number of men who were saved (338). However, the survival rate for men ($338/1690 = 20\%$) was much less than the survival rate for children ($56/109 = 51\%$).