Growing, Growing, Growing: Homework Examples from ACE

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Investigation 1: *Exponential Growth*
ACE #4

Sarah used her calculator to keep track of the number of rubas in Problem 1.2. She found that there will be 2,147,483,648 rubas on square 32.

a. How many rubas will there be on square 33? How many will be on square 34? How many will be on square 35?
b. Which square would have the number of rubas shown here?
2,147,483,648 • 2 • 2 • 2 • 2 • 2 • 2 • 2 • 2 • 2
c. Use your calculator to do the multiplication in part (b). Do you notice anything strange about the answer your calculator gives? Explain.
d. Write 2,147,483,648 • 2 • 2• 2 • 2 • 2 • 2 • 2 • 2 • 2 in scientific notation.
e. Write the numbers 2^{10}, 2^{20}, 2^{30}, and 2^{35} in both standard and scientific notation.
f. Explain how to write a large number in scientific notation.

a. In Problem 1.2 the multiplying factor is 2. We need to take the number of rubas on square 32 and multiply by 2 to get to the number of rubas on square 33, and again by 2 to get to the number of rubas on square 34 etc. Thus the numbers of rubas are respectively: 4,294,967,296; 8,589,934,592; 17,179,869,184.
b. Because this shows the number of rubas on square 32 multiplied by 9 more factors of 2 the result is the number of rubas on square 41.
c. 1.099511628E12. (Students answers will vary depending on the number of digits shown on their calculator screens.)
d. $1.099511628 \times 10^{12}$. (Actually this is not exact, because the calculator screen did not have enough space to show all the digits in the answer. The exact answer would be: $1.099511628576 \times 10^{12}$.)
e. $2^{10} = 1024$ OR 1.024×10^{3}.
$2^{20} = 1048576$ OR 1.048576×10^{6}
f. When you have the number in standard notation you place a decimal point so that the number has a value between 1 and 10. This new number has the same digits as the original number but the place values of the digits have changed. Next you have to choose the correct number of factors of 10 to multiply by to adjust the place values of the digits. For example, if your number in standard form has a leading digit with a place value of “million”, then you will need a 6 factors of 10 in the scientific notation form.

Investigation 1: *Name*
ACE #14

Zak's uncle wants to donate money to Zak's school. He suggests three possible plans. Look for a pattern in each plan.

Plan 1 He will continue the pattern in this table until day 12.

School Donations

Day	1	2	3	4
Donation	\$1	\$2	\$4	\$8

Plan 2 He will continue the pattern in this table until day 10.

School Donations

Day	1	2	3	4
Donation	\$1	\$3	\$9	\$27

Plan 3 He will continue the pattern in this table until day 7.

School Donations

Day	1	2	3	4
Donation	\$1	\$4	\$16	\$64

a. Copy and extend each table to show how much money the school would receive each day.

b. For each plan, write an equation for the relationship between the day number n and the number of dollars donated d.

c. Are any of the relationships in Plans 1, 2, or 3 exponential functions? Explain.

d. Which plan would give the school the greatest total amount of money?

a. It is important that students recognize that these are all exponential patterns: the size of the donation grows by *multiplying* by a given factor each day. The factors are 2, 3, 4 respectively.

Day	1	2	3	....	12
Donation	1	2	4	....	2048

Day	1	2	3	...	10
Donation	1	3	9	...	19683

Day	1	2	3	....	7
Donation	1	4	16	....	4096

b. $d = 2^{n-1}$; $d = 3^{n-1}$; $d = 4^{n-1}$.
Students may have difficulty with the exponent. It is clear that the donations in table 1, for example, are all powers of 2. The problem is that the exponent does not exactly match the day number. Thus, on day 4 the donation is 2^3 or 8, not 2^4. Students met the same pattern when they investigated the "ruba" problem in class.

c. All three plans are exponential, because they can all be written with a growth factor. For plan 1, the growth factor is 2. For plan 2, the growth factor is 3. For plan 3, the growth factor is 4. Students might also determine that they are exponential by looking at the table. Between any two consecutive day values, the donations increase by multiplying the same number each time.

d. Plan 2 would give the most amount of money. This might be surprising, since Plan 3, has the largest growth factor, however, because Plan 3 stops at day 7, it does not have as much of a chance to grow beyond the ten days for Plan 2.

Investigation 1: *Exponential Growth*
ACE #33

Decide whether each number is more or less than one million *without using a calculator* or multiplying. Explain how you found your answer. 3^{10}

This question focuses on the meaning of exponential notation. There are several ways students might think of this. One way is:

3^{10} means 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3 x 3.
One million is 10 x 10 x 10 x 10 x 10 x 10.
If we group 3^{10} as (3 x 3) x (3 x 3) x (3 x 3) x (3 x 3) x (3 x 3), we see that this is the same as 9^5 or 9 x 9 x 9 x 9 x 9.
Now the comparison is clearer:
10 multiplied by itself 6 times is clearly greater than 9 multiplied by itself 5 times.

Checking by calculator: 3^{10} = 59049.

Investigation 2: *Examining Growth Patterns*
ACE #4

As a biology project, Talisha is studying the growth of a beetle population. She starts her experiment with 5 beetles. The next month she counts 15 beetles.

a. Suppose the beetle population is growing linearly. How many beetles can Talisha expect to find after 2, 3, and 4 months?
b. Supose the beetle population is growing exponentially. How many beetles can Talisha expect to find after 2, 3, and 4 months?
c. Write an equation for the number of beetles b after m months if the beetle population is growing linearly. Explain what information the variables and numbers represent.
d. Write an equation for the number of beetles *b* after *m* months if the beetle population is growing exponentially. Explain what the information the variables and numbers represent.
e. How long will it take for the beetle population to reach 200 if it is growing linearly?
f. How long will it take for the beetle population to reach 200 if it is growing exponentially?

This question compares linear and exponential growth. The equations look similar but the operation that shows growth is ADDITION in a linear equation, and MULTIPLICATION in an exponential equation.

a. A linear pattern of growth would mean that the same number of beetles is ADDED every month, in this case 10.

Month, m	0	1	2	3	4
Beetles, b	5	15	25	35	45

b. An exponential pattern of growth would mean that the number of beetles is MULTIPLIED by the same factor every week, in this case 3.

Month, m	0	1	2	3	4
Beetles, b	5	15	45	135	405

c. $b = 5 + 10m$. The "*5*" represents the original number of beetles. The "*10*" represents the rate at which the number of beetles is growing. "1*0m*" represents the number of beetles added to the original *5* after any number, *m*, months.

d. $b = 5(10)^m$. The "*5"* still represents the original number of beetles. The "*10*" represents the growth factor. The "*m"* represents the number of times we have to multiply by *10* to find the number of beetles after any number, *m*, months.

e and f. Students might extend the table to find the answers.

Investigation 3: *Growth Factors and Growth Rates*
ACE #17

Currently 1000 students attend Greenville Middle School. The school can accommodate 1,300 students. The school board estimates that the student population will grow by 5% per year for the next several years.

a. When will the population outgrow the present building?

b. Suppose the school limits growth to 50 students per year. How many years will it take for the population to outgrow the school building?

This question uses both *growth rate* and *growth factor* language. It is quite usual to think of some quantity, for example, a population or a sum of money in a bank account, growing by some *percentage*. The idea *of percentage growth rate* includes the idea of MULTIPLYING, for example by 5%, and then ADDING this amount of growth to the original to produce a new result. You can combine these two operations into one by MULTIPLYING by (1 + 0.05).The multiplication by "1" indicates that the original amount is still there, waiting to be added to the amount of growth.

Note: 1.05 is the *growth factor. 5% is the growth rate.*

a. Students might make a table and extend it until the number of students is greater than 1300. Making the table is more efficient if the growth factor 1.05 is used. (Answers are rounded here)

Year	0	1	2	3	4	5	6
# Students	1000	1000 x 1.05 = 1050	1000 x 1.05^2 = 1103	1158	1216	1276	1340

b. This supposes that the growth is at a constant rate; 50 is ADDED every year. Students might make a table to answer this, or they might write and solve a *linear* inequality: $1000 + 50y > 1300$.

Investigation 4: *Exponential Decay*
ACE #15

Hot coffee is poured in a cup and allowed to cool. The difference between coffee temperature and room temperature is recorded every minute for 10 minutes.

a. Plot the (*time, temperature difference*) data. Explain what the patterns in the table tell you about the rate at which the coffee cools.

Cooling Coffee

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Temperature Difference (°C)	80	72	65	58	52	47	43	38	34	31	28

b. Approximate the decay factor for this relationship.
c. Write an equation for the relationship between time and temperature difference.
d. About how long will it take for the coffee to cool to room temperature? Explain.

It is important to think about the two *variables* here. The second row shows how different the coffee temperature is from the room temperature. The coffee will continue to cool until this difference is zero.

a. The plot is not shown here. The pattern of change shown in the table will reflect in the shape of the graph, that is, the temperature falls fastest to begin with, so the difference between coffee and room temps will change faster to begin with; but, as the two temps get more alike, the rate of cooling will slow. The graph should have a steep slope for low values of time, but a less steep slope as time passes.

b. To approximate the decay factor we look for a number that multiplies each temperature difference to get the next temperature difference. We can get this by looking at the ratios 72/80, 65/72, 58/65, 52/58, 47/52 etc. These ratios are 0.9, 0.903, 0.94, 0.90, 0.904 etc. It looks like 0.9 would be a good approximation of the decay factor. **Note: this is the same as saying that the temp difference decreases by 10% each time.**

c. $D = 80(0.90)^T$

d. Students can continue the table until the difference is zero.

Investigation 4: *Exponential Decay*
ACE #17

Answers parts (a) and (b) without using your calculator.

a. Which decay factor represents faster decay, 0.8 or 0.9?
b. Order the following from least to greatest:
   0.9^4 0.9^2 90% $^2/_{10}$ $^2/_9$ 0.8^4 0.84

This question goes to the heart of what a *decay factor* means. If we multiply a quantity by 0.8 then the result is that we lose 20% of the original quantity, and retain 80% of the original quantity.
If the decay factor is 0.9 then we lose 10% of the original quantity. Thus a decay factor of 0.8 loses more of the original quantity than a decay factor of 0.9.

a. 0.8 represents faster decay than 0.9.

b. Multiplying anything by 0.9 loses 10% of the original quantity. Thus repeatedly multiplying by 0.9 keeps making the quantity smaller. Thus, 0.9^4 is less than 0.9^2, which is less than 0.9 or 90%.
$^2/_{10}$ is 0.2, which seems clearly smaller than 0.9^4 and 0.9^2 and 0.9. Students should also be able to compare $^2/_{10}$ and $^2/_9$ without using a calculator. They should also be able to compare 0.8^4 and 0.9^4. They should be able to mentally compute $0.9^2 = 0.81 < 0.84$. Given all these comparisons they should be able to arrange all the quantities:
$^2/_{10} < {}^2/_9 < 0.8^4 < 0.9^4$ …

Investigation 5: *Patterns with Exponents*
ACE #11

Tell whether the statement is *true* or *false*. Explain.

$\frac{5^{12}}{5^4} = 5^3$

This question gets to the meaning of exponential form.
5^{12} means 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5 x 5.
5^4 means 5 x 5 x 5 x 5.

So, $\frac{5^{12}}{5^4} = \frac{(5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5)}{(5 \times 5 \times 5 \times 5)}$

We can then rewrite this fraction in an equivalent form by dividing the same factor, 5 x 5 x 5 x 5, out of the numerator and denominator, to get:

$$\frac{5^{12}}{5^4} = \frac{(5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5)}{1} = 5^8$$

Investigation 5: *Patterns with Exponents*
ACE #66

Grandville has a population of 1,000. Its population is expected to decrease by 4% a year for the next several years. Tinytown has a population of 100. Its population is expected to increase by 4% a year for the next several years.

a) What is the population of each town after 5.5 years?
b) In how many years will Tinytown have a population of approximately 1,342? Explain your method.
c) Will the populations of the two towns ever be the same? Explain.

This asks students think about 2 different exponential relationships: the first is an exponential decay relationship, modeling a 4% loss each year, $P = 1000(0.96)^y$; and the second is an exponential growth relationship modeling a 4% increase every year, $P = 100(1.04)^y$. Students should be able to picture these two relationships as graphs, without actually using a calculator. The first "starts" at P = 1000 and decreases; the second "starts" at P = 100 and increases. The curves will eventually cross.

a) To find the population of Grandville after 5.5 years, substitute 5.5 in for y, in the equation: $P = 1000(0.96)^y$. Doing this gives the equation $1000(0.96)^{5.5} \approx 799$. Similarly, to find the population of Tinytown substitute 5.5 in the equation $P = 1000(1.04)^y$, resulting in the equation $1000(1.04)^{5.5} \approx 124$.

b) Around 66 years: $100 \times 1.04^{66} \approx 1331$. Students might try to do some sort of guess-and-check method also using the "Rule of 72" from Problem 3.3. The "Rule of 72" is a way to approximate how long something takes to double if the relationship grows exponentially. Since the growth rate is 4%, we have the equation: $72 \div 4 = 18$, which gives the following rough approximations:

$1.04^{18} \approx 2$,
So…
$100 \times 1.04^{18} \approx 200$,
$100 \times 1.04^{36} = 100 \times 1.04^{18} \times 1.04^{18} \approx 400$,
$100 \times 1.04^{54} = 100 \times 1.04^{18} \times 1.04^{18} \times 1.04^{18} \approx 800$,
$100 \times 1.04^{72} = 100 \times 1.04^{18} \times 1.04^{18} \times 1.04^{18} \times 1.04^{18} \approx 1600$,

Since 1342 is between 800 and 1600, it follows that the number of years should be somewhere between 54 and 72.

c) Yes; the two towns will have the same populations if they continue to change at the same rates. Even though Grandville has a greater starting population, its population is decreasing, while Tinytown's population is increasing. So, eventually the graphs will cross. However, it will take about 28 years for this to happen.