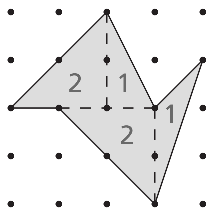
Mathematics Background

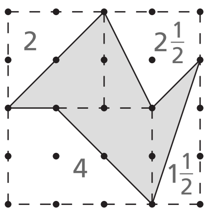
Finding Area and Distance

Students' work in this Unit develops a fundamentally important relationship connecting geometry and algebra: the Pythagorean Theorem. The presentation of ideas in the Unit reflects the historical development of the concept of irrational numbers. The need for such numbers was recognized by early Greek mathematicians as they searched for ratios of integers to represent side lengths of squares with certain given areas, such as 2 square units. The square root of 2 is an irrational number, which means that it cannot be written as a ratio of two integers.

Students find areas of plane figures drawn on dot grids. This activity reviews some concepts developed in the Grade 6 unit *Covering and Surrounding*. One common method for calculating the area of a figure is to subdivide it and add the areas of the component shapes. A second common method is to enclose the shape in a rectangle and subtract the areas of the shapes that lie outside the figure from the area of the rectangle. Below, the area of the shape is found with each method.

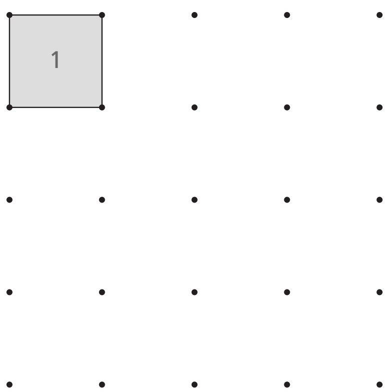


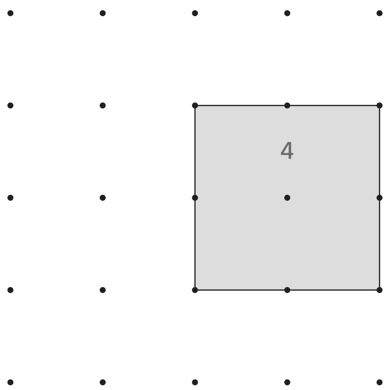
Subdivide to find the area:
$2 + 2 + 1 + 1 = 6$



Enclose in a square to find the area:
$16 - (4 + 2 + 2\frac{1}{2} + 1\frac{1}{2}) = 6$

In Investigation 2, students draw squares with as many different areas as possible on a 5 dot-by-5 dot grid. There are eight possible squares, four "upright" and four "tilted." Visit Teacher Place at mathdashboard.com/cmp3 to see the image gallery.





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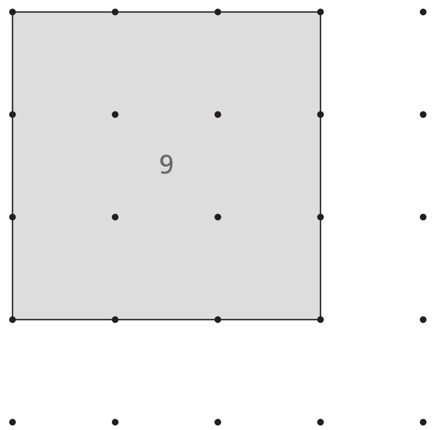
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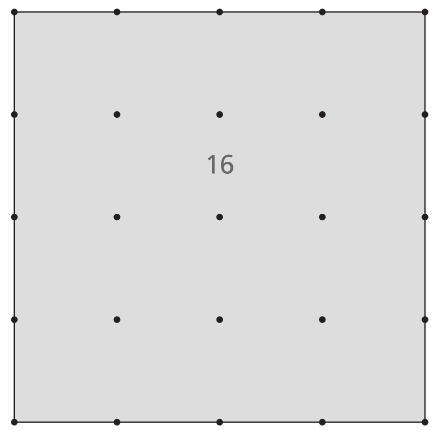
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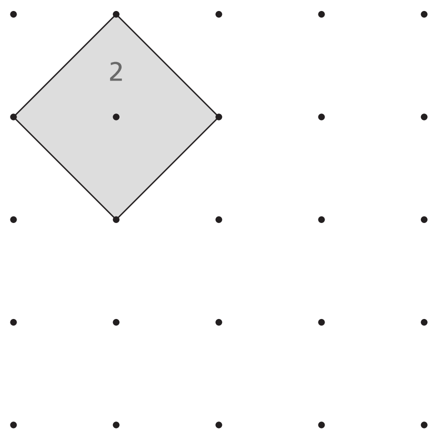
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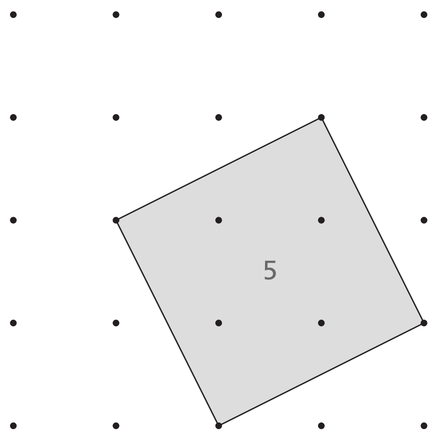
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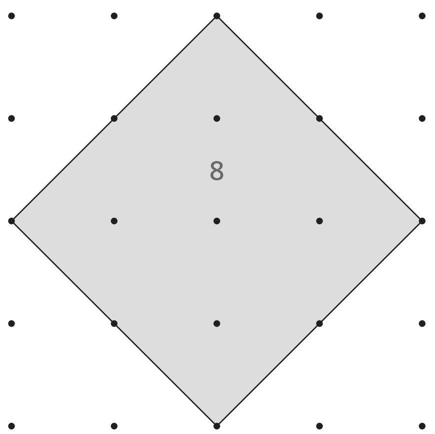
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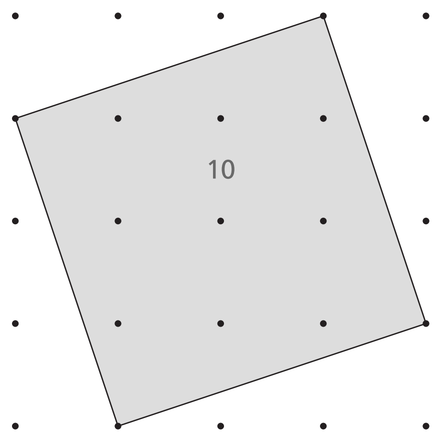
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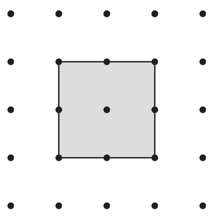
Square Roots

If the area of a square is known, its side length is easy to determine: It is the number with a square equal to the area. The lengths of the sides of the preceding squares (in units) are 1, 2, 3, 4, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, and $\sqrt{10}$. The fact that some of these lengths are not whole numbers prompts the introduction of the $\sqrt{\ }$ symbol. Because the grid is a centimeter grid, students can estimate the values of the square roots by measuring these lengths with a ruler.



This square has an area of 4 square units. The length of each side is the square root of 4 units, which is equal to 2 units.

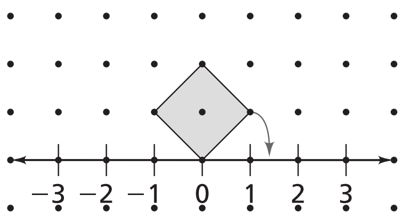
By making these ruler estimates and comparing them to estimates obtained by computing square roots on a calculator, students develop a sense of these numbers and begin to realize that they cannot be expressed as terminating or repeating decimals.

Students also develop benchmarks for estimating square roots. For example, you can estimate $\sqrt{5}$ as follows.

- $\sqrt{5}$ is between 2 and 3 because $2^2 < 5 < 3^2$.
- Since 5 is closer to 4 than to 9, you can estimate that $\sqrt{5}$ is closer to 2 than to 3.
- Try 2.25: $2.25^2 = 5.06$.
- So, $\sqrt{5}$ is between 2 and 2.25 but closer to 2.25.
- Try 2.24: $2.24^2 = 5.0176$, which is closer.

You can continue this method until the desired accuracy is obtained.

Students also estimate square roots with a number line ruler, which helps them develop a sense of the size of the irrational numbers, such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$. One way to locate $\sqrt{2}$ on the number line is as follows.



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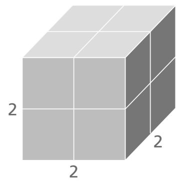
The square above has an area of 2 square units. The length of a side of this square is $\sqrt{2}$ units. If you draw a number line as shown, and use a compass to mark off a segment with the same length as a side of the square, you can see that the segment is about 1.4 units long.

Cube Roots

If the volume of a cube is known, its edge length is easy to determine: It is the number that is raised to the third power to give the volume. Students might be familiar with some perfect cubes, such as 1, 8, 27, and 64.



This cube has the volume of 8 cubic units. The length of each edge is the cube root of 8 units, which is equal to 2 units.

Not all volumes produce whole numbers, though, so this prompts the introduction of the $\sqrt[3]{}$ symbol.

Students also develop benchmarks for estimating cube roots. For example, you can estimate $\sqrt[3]{10}$ as follows.

- $\sqrt[3]{10}$ is between 2 and 3 because $2^3 < 10 < 3^3$.
- Since 10 is closer to 8 than to 27, you can estimate that $\sqrt[3]{10}$ is closer to 2 than to 3.
- Try 2.2: $2.2^3 = 10.648$.
- So, $\sqrt[3]{10}$ is between 2 and 2.2 but closer to 2.2.
- Try 2.15: $2.15^3 \approx 9.938$, which is closer.

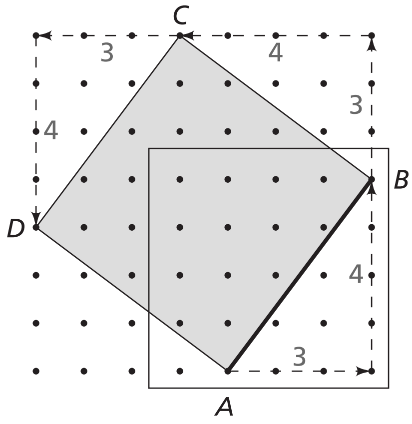
You can continue this method until the desired accuracy is obtained.

Note: If $x^3 < y < z^3$, then $x < \sqrt[3]{y} < z$. However, be wary of deciding whether $\sqrt[3]{y}$ is closer to x or to z. For example, $2^3 < 2.51^3 < 3^3$. Although $2.51^3 \approx 15.813$ is closer to 2^3 than to 3^3, 2.51 is closer to 3 than to 2.

Using Square to Find Lengths of Segments

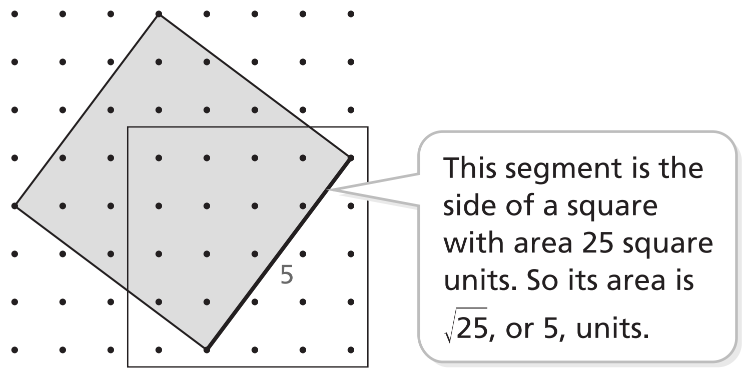
Finding the areas of squares leads students to a method for finding the distance between two dots. The distance between two dots on a dot grid is the length of the line segment connecting them. To find this length, students can draw a square with the segment as one side. The distance between the two dots is the square root of the area of the square.

To use this method to find all the different lengths of segments that can be drawn on a 5 dot-by-5 dot grid, the grid must be extended to fit the squares associated with those lengths.



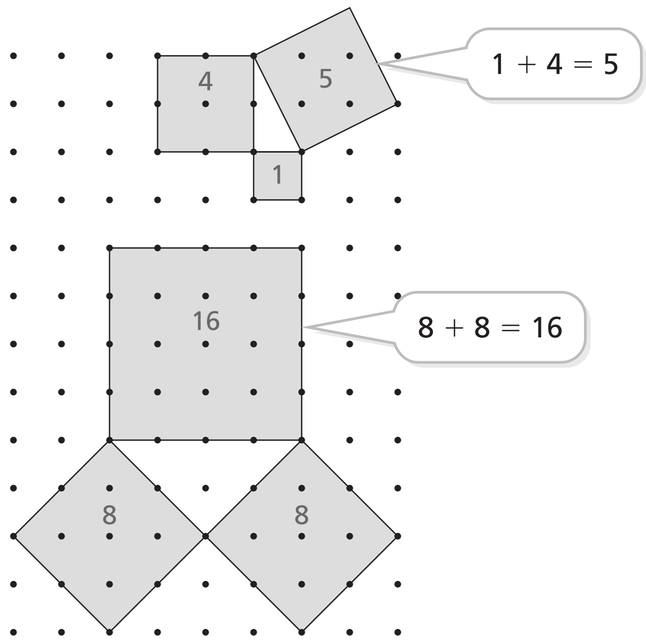
To draw the square with the given side length, many students will use an "up and over" or "down and over" method to go from one point to the next. For example, to get from Point *A* to Point *B*, you go over 3 units and up 4 units. These points are two vertices of the square. To get to the third vertex *C*, go up 3 units and over 4 units. To get the fourth vertex *D*, go over 3 units and down 4 units. In this way, students are developing intuition about the Pythagorean Theorem.

Having found the four corners, students can draw the square shown below. An efficient way to find its area is to note that the unshaded portion of the 7-by-7 grid is occupied by four congruent triangles, each with a base of 3 units and a height of 4 units. The combined area of the triangles is therefore 24 square units, and the remaining area of the grid must be $7^2 - 24$, or 25 square units. Whatever method students use, they should find the tilted square has an area of 25 square units. Its side length must be 5 units.

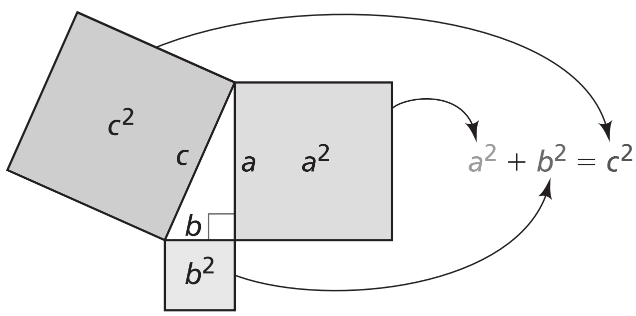


Developing and Using the Pythagorean Theorem

Once students are comfortable with finding the length of a segment by thinking of it as the side of a square, they investigate the patterns among the areas of the three squares that can be drawn on the sides of a right triangle.



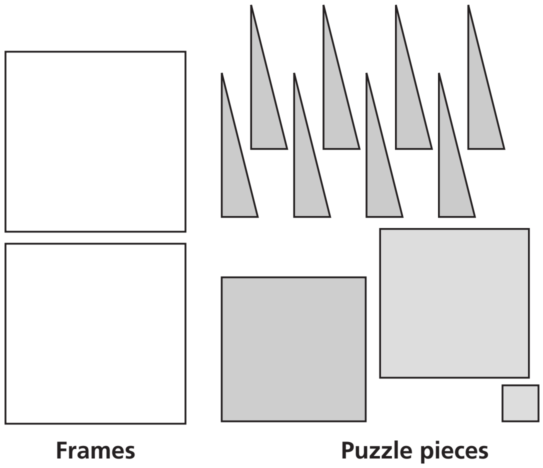
The observation that the square on the hypotenuse has an area equal to the sum of the areas of the squares on the legs leads students to discover the Pythagorean Theorem: If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.



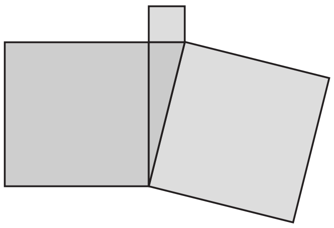
A *theorem* is a general mathematical statement that has been proven true. More than 300 different proofs have been devised for the Pythagorean Theorem. It is regarded as one of the most important developments in mathematics, because it allows us to link ideas of number to ideas of space.

A Proof of the Pythagorean Theorem

Students solve a puzzle that gives a geometric proof of the Pythagorean Theorem. The puzzle pieces consist of eight congruent right triangles and three squares.



The side lengths of the squares are the lengths of the three triangle sides.



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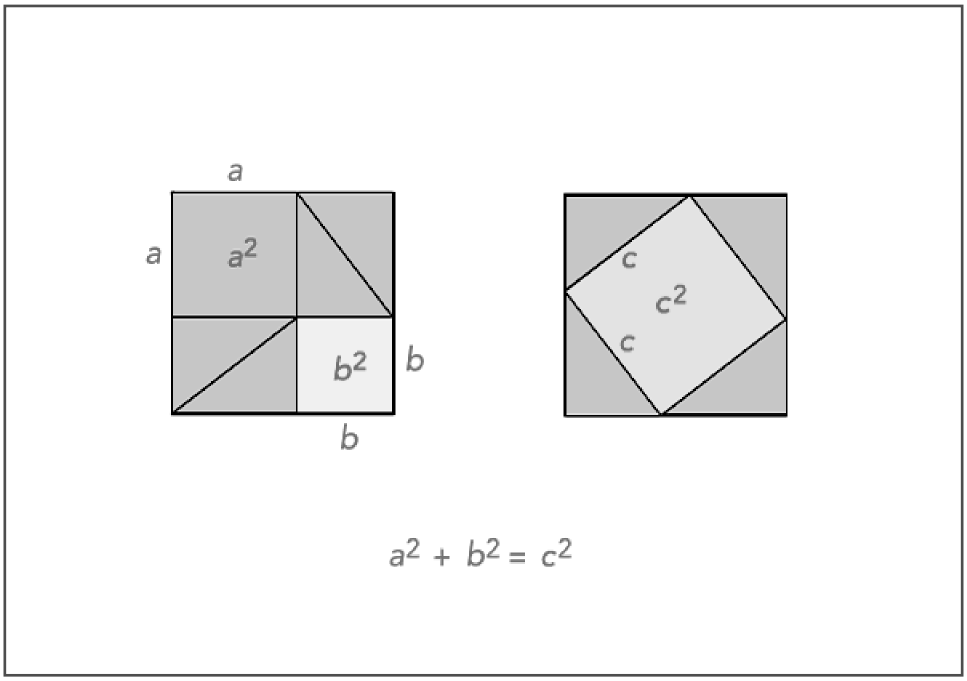
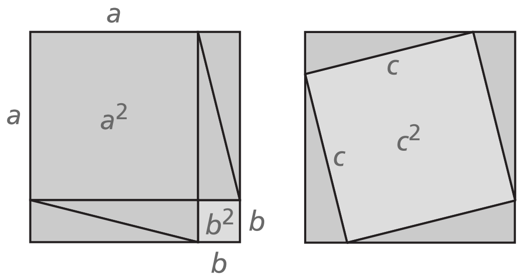
To solve the puzzle, students must arrange the pieces to fit into two square puzzle frames. Students' arrangements of the 11 shapes may differ slightly, but all arrangements lead to the same conclusion. Visit Teacher Place at mathdashboard.com/cmp3 to see the complete video.



Geometrically, the diagram shows that if the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$.

You can make similar puzzle pieces starting with any right triangle and then arrange the shapes in the same way. Therefore, the statement $a^2 + b^2 = c^2$ is true for any right triangle.

In later courses, students may see this geometric argument presented algebraically. The sum of the areas of the two squares and the four triangles in the left frame equals the sum of the areas of the square and the four triangles in the right frame.

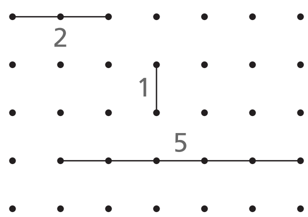


$$a^2 + b^2 + 4\left(\frac{ab}{2}\right) = c^2 + 4\left(\frac{ab}{2}\right)$$
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem has many applications that connect the concepts of line segment lengths, squares, and right angles.

Using the Pythagorean Theorem to Find Lengths

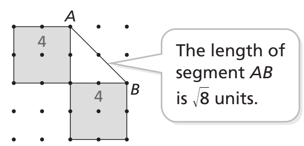
Students use the Pythagorean Theorem to find the distance between two dots on a dot grid. The length of a horizontal or vertical segment drawn on a dot grid can be found by counting the units directly, as shown below.



If the segment is not vertical or horizontal, you can treat it as the hypotenuse of a right triangle with vertical and horizontal legs. You can find the length of the hypotenuse—and thus the distance between the dots—with the Pythagorean Theorem.

To find the length of line segment *AB* below:

- Draw a right triangle with segment *AB* as the hypotenuse.
- Calculate the areas of the squares on the legs of the triangle (4 square units each).
- Add these areas (8 square units, which is the area of the square drawn on the hypotenuse).
- Take the square root of this sum.



The Converse of the Pythagorean Theorem

The converse of a statement of the form "If *p* then *q*" is "If *q* then *p*." The converse of the Pythagorean Theorem states: If *a*, *b*, and *c* are the lengths of the sides of a triangle such that $a^2 + b^2 = c^2$, then the triangle is a right triangle.

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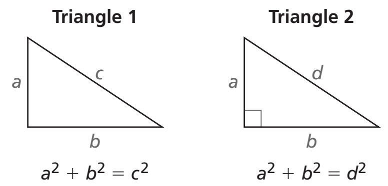
The converse of a true statement is not always true. (For example, consider the true statement "If a figure is a square, then it is a rectangle." Its converse, "If a figure is a rectangle, then it is a square," is false.) The converse of the Pythagorean Theorem, however, is true and can be used to show that a given triangle is a right triangle. For example, if you know the side lengths of a triangle are 6 in., 8 in., and 10 in., then because $6^2 + 8^2 = 10^2$, you can conclude that the triangle is a right triangle.

> You are given Triangle 1 with side lengths of a, b, and c, where $a^2 + b^2 = c^2$. You want to prove that Triangle 1 is a right triangle. To do so, draw a right triangle, Triangle 2, with legs of lengths a and b and a hypotenuse of length d. Then $a^2 + b^2 = d^2$ by the Pythagorean Theorem.



> You know that $a^2 + b^2 = c^2$ (given). By substitution, $c^2 = d^2$, so $c = d$. Triangle 1 and Triangle 2 have congruent corresponding sides and are therefore congruent. So, Triangle 1 is a right triangle.

In this Unit, students build triangles with a variety of different side lengths and determine whether they are right triangles. Based on their findings, they conjecture that triangles with side lengths that satisfy $a^2 + b^2 = c^2$ are right triangles. They then examine an informal proof of the converse.

An interesting byproduct of the converse of the Pythagorean Theorem is the concept of *Pythagorean triples*, sets of numbers that satisfy the relationship $a^2 + b^2 = c^2$. Students discover that finding Pythagorean triples means finding two square numbers with a sum that is also a square number.

Multiples of one triple will generate countless others. For example, once you establish that 3-4-5 is a Pythagorean triple, you know that 6-8-10, 9-12-15, 12-16-20, and so on, are also Pythagorean triples. You can use similarity to deduce that any scale copy of a triangle known to be a right triangle will also be a right triangle. Or you can use algebra to show that if $a^2 + b^2 = c^2$, then $(ka)^2 + (kb)^2 = (kc)^2$.

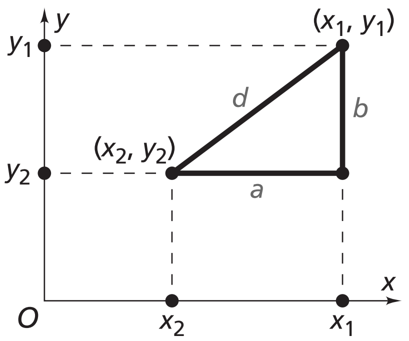
Distance Formula

To find the distance between two points, you can use the distance formula.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The distance formula is simply a different form of the Pythagorean Theorem, $a^2 + b^2 = c^2$.

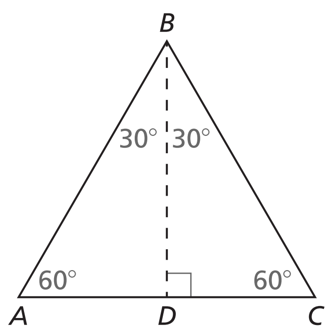
The diagram below shows two points, (x_1, y_1) and (x_2, y_2). Leg a is the horizontal distance, $x_1 - x_2$, between the two points. Leg b is the vertical distance, $y_1 - y_2$, between the points. The hypotenuse represents the distance between the two points. Thus, $a^2 + b^2 = c^2$ becomes $(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$. Taking the square root of both sides yields the distance formula.



Students will learn about the distance formula in high school.

Special Right Triangles

In Investigation 5, students learn about 30-60-90 triangles by starting with an equilateral triangle (a 60-60-60 triangle). They fold the triangle in half, forming two congruent 30-60-90 triangles. For each of these triangles, they deduce that the leg opposite the 30° angle is half the length of the side of the original triangle. That is, the lengths of segments *AD* and *CD* are half the length of segment *AC*.



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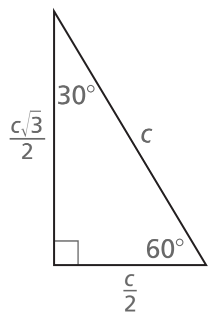
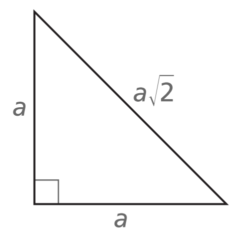
Suppose the hypotenuse of a 30-60-90 triangle has length c. The length of the side opposite the 30° angle must be half this length, or $\frac{c}{2}$. Using the Pythagorean Theorem, the square of the length of the longer leg is $c^2 - \left(\frac{c}{2}\right)^2 = c^2 - \frac{c^2}{4}$, or $\frac{3c^2}{4}$. So, its length is $\sqrt{\frac{3c^2}{4}}$ or $\frac{c\sqrt{3}}{2}$.

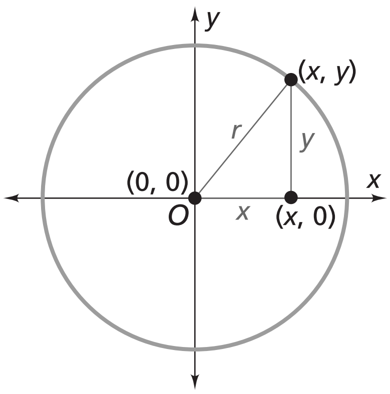


Students also explore isosceles right triangles (45-45-90 triangles) in the ACE Exercises, and find that the length of the hypotenuse is always the length of one of the legs times $\sqrt{2}$. If the length of each leg is a, then by the Pythagorean Theorem, the square of the length of the hypotenuse must be $a^2 + a^2$, or $2a^2$. Therefore, the length of the hypotenuse is $\sqrt{2a^2} = a\sqrt{2}$.



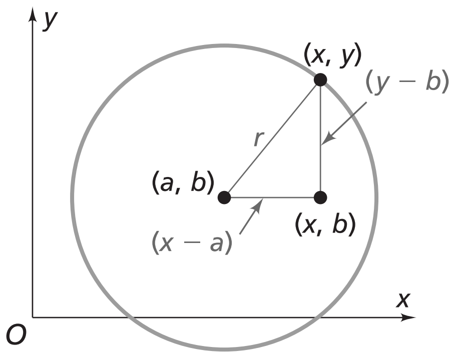
Equation of a Circle

You can define a circle as a set of points equidistant from a fixed point called the *center* of the circle. When a circle of radius r is drawn on a coordinate grid with center at the origin, any point on the circle (x, y) can be viewed as a vertex of a right triangle with legs of length x and y and hypotenuse of length r.



Applying the Pythagorean Theorem to the right triangle, you have $x^2 + y^2 = r^2$. This equation would be true for any point (x, y) on the circle. So, the equation represents the relationship for all points on the circle.

For circles centered at points other than the origin, such as (h, k), the equation becomes $(x - h)^2 + (y - k)^2 = r^2$.



The more general equation for a circle can also be generated from the distance formula. Using the distance formula, the distance from the center (h, k) to a point (x, y) on the circle is:

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

where r is radius of the circle. Squaring both sides of the equation yields:

$$r^2 = (x - h)^2 + (y - k)^2$$

Rational and Irrational Numbers

When you examine patterns in the decimal representations of fractions, or rational numbers, you find that the decimals either terminate or repeat. In fact, since one way of interpreting any fraction $\frac{a}{b}$ is as $a \div b$, you can deduce that any long division by b has a finite number of remainder possibilities before one must repeat or the process terminates with a remainder of zero.

$\frac{1}{5} = 0.2$	Terminating decimal
$\frac{1}{3} = 0.3333333...$	Repeating decimal
$\sqrt{2} = 1.414...$	Nonterminating, nonrepeating decimal

Numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be expressed as repeating or terminating decimals. Students create line segments with these lengths. For example, $\sqrt{2}$ is the length of the hypotenuse of a right triangle with legs of length 1. They then locate the lengths on a number line. This procedure helps students to estimate the size of these irrational numbers.

Converting Repeating Decimals to Fractions

Because all repeating decimals are rational numbers, they can be represented as fractions. It is not always obvious, though, what fraction is equivalent to a given repeating decimal. One method for converting a repeating decimal to a fraction involves solving an equation. For example, you can convert 12.312312 . . . as follows.

$$\text{Let } N = 12.312312\ldots$$

$$1{,}000N = 12{,}312.312312\ldots \quad \text{Multiply each side by 1,000.}$$

$$-\quad N = 12.312312\ldots \quad \text{Subtract the first equation from the second.}$$

$$999N = 12{,}300$$

$$\frac{999N}{999} = \frac{12{,}300}{999} \quad \text{Divide each side by 999.}$$

$$N = 12\frac{312}{999} \quad \text{Simplify.}$$

The decimal equivalents of fractions with denominators of 9, 99, 999, and so on, display interesting patterns that can be used to write repeating decimals as fractions. For example, all decimals with a repeating part of one digit, such as 0.111 . . . and 0.222 . . . can be written as a fraction with 9 in the denominator and the repeated digit in the numerator, such as $\frac{1}{9}$ and $\frac{2}{9}$. Decimals with a repeating part of two digits, such as 0.010101 . . . and 0.121212 . . . can be written as a fraction with 99 in the denominator and the repeated digits in the numerator, such as $\frac{1}{99}$ and $\frac{2}{99}$.

Fraction	Decimal	Fraction	Decimal
$\frac{1}{9}$	0.1111 . . .	$\frac{1}{11}$	0.0909 . . .
$\frac{2}{9}$	0.2222 . . .	$\frac{2}{11}$	0.1818 . . .
$\frac{3}{9}$	0.3333 . . .	$\frac{3}{11}$	0.2727 . . .
$\frac{4}{9}$	0.4444 . . .	$\frac{4}{11}$	0.3636 . . .
$\frac{5}{9}$	0.5555 . . .	$\frac{5}{11}$	0.4545 . . .
$\frac{6}{9}$	0.6666 . . .	$\frac{6}{11}$	0.5454 . . .
$\frac{7}{9}$	0.7777 . . .	$\frac{7}{11}$	0.6363 . . .

Proof that the Square Root of 2 is Irrational

In high school, students may prove that $\sqrt{2}$ is not a rational number. Its irrationality can be proved in an interesting way—a proof by contradiction. The proof is given here for the teacher's information.

Assume $\sqrt{2}$ is rational. Then, there exist positive integers p and q such that $\sqrt{2} = \frac{p}{q}$, where $q \neq 0$. So, $\sqrt{2}q = p$. Squaring both sides gives $2q^2 = p^2$. From the *Prime Time* unit students learned that all square numbers have an odd number of factors. The reason is that factors of a number come in pairs. In a square number the factors in one of the pairs must be equal, which makes the number of factors for a square number odd. This means that if p and q are positive integers, then p^2 and q^2 both have an odd number of factors. Since $p^2 = 2q^2$, p^2 has the same number of factors as $2q^2$. But $2q^2$ has an even number of factors (The factor 2 plus the odd number of factors of q^2.) This is a contradiction. Therefore p and q cannot exist with these properties and $\sqrt{2}$ must be irrational.

Square Root Versus Decimal Approximation

Problems involving the Pythagorean Theorem often result in square roots that are irrational numbers. Students at this level are often reluctant to leave numbers in a square root form. For example, rather than give an answer of $\sqrt{3}$, they give a decimal approximation, such as 1.732.

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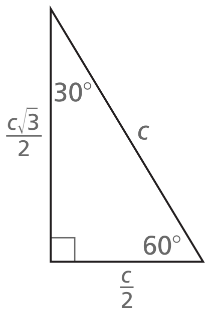
Some students are not comfortable thinking about square roots as numbers. Although it is important to know the approximate size of an answer, especially in a practical problem, it is sometimes better to give an exact answer, and this often means using square root form. For example, in the study of 30-60-90 triangles,

$$\frac{\text{length of leg opposite the }60^\circ\text{angle}}{\text{length of the hypotenuse}} = \frac{\sqrt{3}}{2}$$



Here, $\sqrt{3}$ is much easier to remember than a multidigit decimal approximation, and the expression using the square root gives the exact result.

Similarly, in a right triangle with hypotenuse of length 9 units and one leg with length 8 units, the length of the other leg is $\sqrt{81 - 64}$, or $\sqrt{17}$ units. This answer is exact, while the calculator answer, 4.123105626, is an approximation.

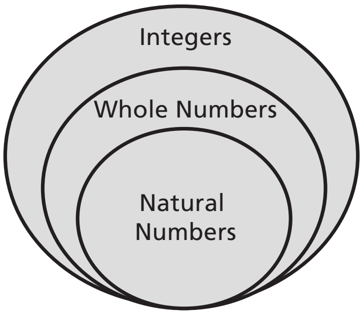
This is not to say that all answers should be left in square root form—context needs to be considered. Heights of buildings are more easily comprehended in whole-number or decimal form, even if that form does not give the precise answer. Students should be encouraged to leave an answer in square root form when there is no practical reason to express it as a decimal approximation. The hope is that all students will become comfortable with square roots as numbers in contexts where expressing an answer as a square root is appropriate. In this Unit, students will develop a "sense" of square roots as numbers and some idea of where they fit on the number line (between what two rational numbers they occur).

When working with measured quantities, you should discuss what is a reasonable level of accuracy for the answer. The level of accuracy will depend on the context and the measuring tools.

Number Systems

New number systems are created when a problem arises that cannot be answered within the system currently in use, or when inconsistencies arise that can be taken care of only by expanding the domain of numbers in the system.

The historical "discoveries" of new number systems in response to needs are reflected in the number sets students use in grades K–12. Elementary students begin with the *natural numbers*, also called *counting numbers*. Then, zero is added to the system to create the set of *whole numbers*. Later, students learn that negative numbers are needed to give meaning in certain contexts, such as temperature. Now they have the number system called the *integers*, shown below.



In elementary and middle school, students learn about fractions and situations in which fractions are useful, as in many division problems. Students' number world has been expanded to the set of *rational numbers*.

In this Unit, students encounter contexts in which the need for *irrational numbers* arises. Specifically, they need irrational numbers to express the exact lengths of tilted segments on a grid. The set of rational numbers and the set of irrational numbers compose the set of *real numbers*. The diagram below represents all these sets of numbers.