Mathematics Background

Making Sense of Symbols

Students come to this Unit with considerable experience working with symbolic expressions, tables, and graphs arising from their study of mathematics in Problems from prior Units. Beginning in Grade 6, students learn to examine multiple representations (equations, graphs, tables, diagrams, and verbal descriptions) to help them understand mathematical relationships and to represent mathematical relationships in a variety of ways. In Grade 6, students develop formulas for the area and perimeter of two-dimensional shapes by looking at the relationship between the dimensions of a shape and its perimeter or area. They are introduced to the basic language, concepts, and representations of algebra in *Variables and Patterns*.

In Grade 7, students study geometric relationships, such as the relationship between the number of sides and measures of angles of regular polygons. They also study numerical relationships, such as finding a missing factor or addend in a number sentence. In *Moving Straight Ahead*, they examine linear models in detail, including patterns of change that characterize linear relationships, and they solve equations using tables, graphs, and symbolic methods.

In Grade 8, *Thinking With Mathematical Models* continues the study of linear relationships, but shifts the focus to examining and comparing linear and nonlinear functions, in particular, inverse variation relationships. *Looking For Pythagoras* provides a geometric interpretation of the Pythagorean Theorem. It also provides a geometric interpretation of parallel and perpendicular lines. Questions about variables, relationships, patterns of change, and representations are raised in *Moving Straight Ahead* and applied to exponential functions in *Growing, Growing, Growing*. Quadratic relationships are studied and applied in *Frogs, Fleas, and Painted Cubes*. Up to this point in the development of algebra, representing patterns and reasoning about patterns of change using multiple representations has been the main focus. In addition, students have studied the properties of real numbers, such as the Commutative and Distributive properties, first in *Prime Time*. They have used these properties throughout the remaining Units.

In this Unit, *Say It With Symbols*, the emphasis shifts to using the properties of numbers to look at equivalent expressions and the information each expression represents in a given context and to interpreting the underlying patterns that a symbolic equation or statement represents. Students look critically at each part of an expression and how each part relates to the original expression. They examine the graph and table of an expression as well as the context the expression is modeling. The properties of equality and numbers are used extensively in this Unit as students write and interpret equivalent expressions, combine expressions to form new ones, predict patterns of change represented by an equation or expression, and solve equations. Students continue to develop their algebraic skills in the remaining Grade 8 and Algebra 1 Units. Collectively, the Algebra Units and, in particular, this Unit help to develop what we call “symbol sense.”

This Unit develops students' facility in reasoning with purely symbolic expressions. They observe how symbolic expressions are used in real situations in the Problems. This Unit provides meaningful settings to motivate conventional algebraic notation and techniques. Thinking with symbolic expressions, in situations where mathematics is applied, plays a significant role in developing a student's "symbolic sense" or fluency with symbolic statements.

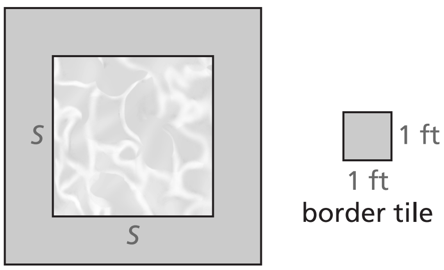
The Unit is organized around five aspects of symbolic expressions: creating and interpreting equivalent expressions, combining expressions, solving equations, observing patterns of change, and reasoning with symbols. Throughout all of the Investigations, the Problems require students to write symbolic statements to model a situation, interpret symbolic statements, write equivalent symbolic expressions, and make predictions using symbolic statements.

Equivalent Expressions

In *Variables and Patterns* and *Moving Straight Ahead*, students explored ways in which relationships can be expressed in tables, graphs, and equations. The contextual clues or the patterns in tables or graphs strongly influenced the construction of a single equation or expression, so students did not gain experience with equivalent equations. In this Unit, students are deliberately presented with situations in which contextual clues can be interpreted in several ways to produce different equations or expressions that are equivalent.

Example

Find the number of 1-foot-square tiles N needed to make a border around a square pool with sides of length s feet.



Different conceptualizations of the situation can lead to different equivalent expressions for the number of tiles; e.g., $N = 4(s + 1)$, $N = 4(s + 2) - 4$, or $N = (s + 2)^2 - s^2$.

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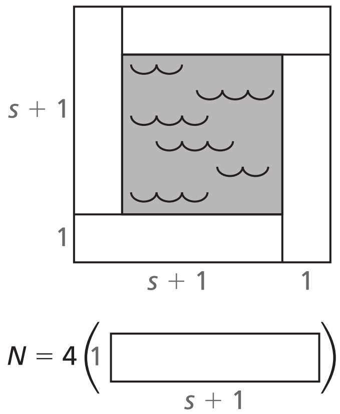
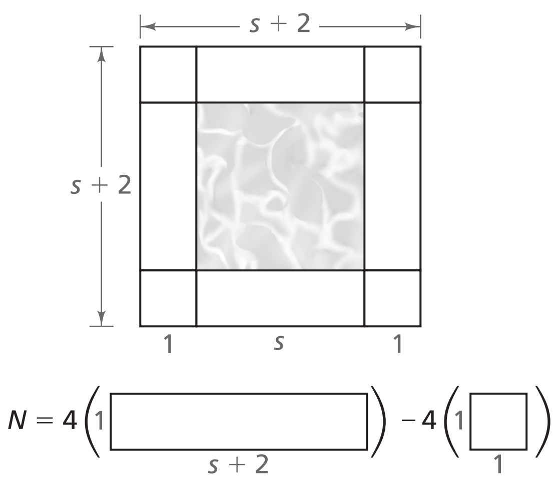
Verifying Equivalence

At this stage of development, students may consider the reasonableness of the geometric reasoning represented by each equation. For example, the equation $N = 4(s + 1)$ represents the following geometric pattern: The border is divided into four rectangles with dimensions $(s + 1)$ and 1, resulting in the equation $N = 4(s + 1)$.



Or students might use the following geometric pattern for the equivalent equation $N = 4(s + 2) - 4$.

> **Example**
> Add the four long strips along the sides and then subtract the four corner tiles that are counted twice, resulting in the equation $N = 4(s + 2) - 4$.



Students may also generate a table or graph to show that the expressions are equivalent for the number of border tiles. Some students may realize from the table and graph that the relationship is linear. They find that the constant rate of change is 4, the y-intercept is 4, and the equation is $N = 4s + 4$.

In Problem 1.2, students also verify that the expressions for the number of border tiles are equivalent using the Distributive and Commutative properties. They may have questions about the expression $(s + 2)^2 - s^2$ for the number of border tiles because it seems quadratic. Using the Distributive and Commutative properties, they verify that it is equivalent to $4s + 4$.

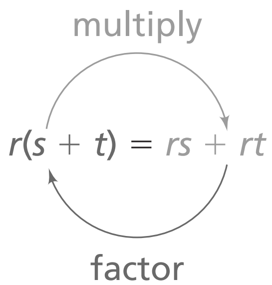
The Distributive Property

The Distributive Property was first introduced in *Prime Time* and then used in several sixth, seventh, and eighth grade Units in the linear form, $a(b + c) = ab + ac$. In *Frogs, Fleas, and Painted Cubes*, the Distributive Property was extended to binomials, $(a + b)(c + d)$. In each case, an area model was used to show the relationship between the expanded and factored form of an expression. The diagram below illustrates two aspects of the Distributive Property. If an expression is written as a factor multiplied by a sum of two or more terms, the Distributive Property can be applied to multiply the factor by each term in the sum. If an expression is written as a sum of terms, and the terms have a common factor, the Distributive Property can be applied to rewrite or factor the expression as the common factor multiplied by a sum of two or more terms.

The Distributive Property

$r(s + t) = rs + rt$,

for any real numbers r, s, and t.



The Distributive Property allows students to group symbols or to expand an expression. It is one of the most important properties for writing equivalent expressions. In Investigation 3, a realistic context motivates a rule for distributing a negative sign. The following example provides some informal understanding for the general idea that $a - (b + c) = a - b - c$.

> **Example**
>
> Suppose a checking account contains \$100 at the start of the week. Two checks are written during the week, one for \$22 and one for \$50. Find the balance in the account at the end of the week.
>
> **Method 1**
>
> $100 - (22 + 50) = 28$
>
> **Method 2**
>
> $100 - 22 - 50 = 28$

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Mathematically, the relationship can be represented as an equation and rewritten.

$$\begin{aligned} a - (b + c) &= a + (-1)(b + c) \\ &= a + (-1)b + (-1)c \\ &= a - b - c \end{aligned}$$

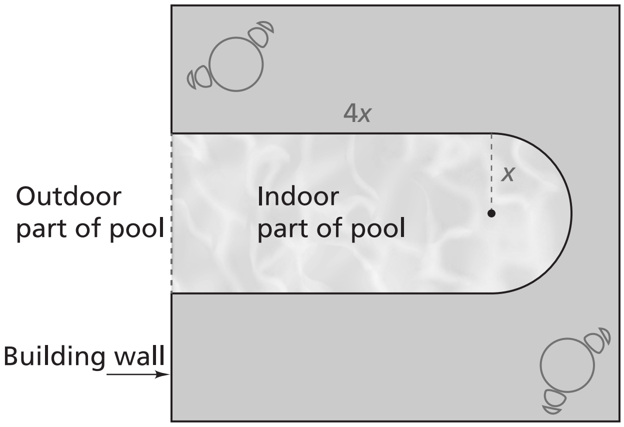
Practice multiplying binomials and factoring simple quadratic expressions is provided throughout the Unit as students write equivalent quadratic expressions or solve quadratic equations.

Interpreting Expressions

Identifying and interpreting information that is represented by expressions occurs in all of the Problems. In the first two Problems, 1.1 and 1.2, students interpret the geometric patterns that are represented by each student's equation for the number of border tiles based on the dimensions of the square pool. In Problem 1.3, students interpret a symbolic expression that has the added complexity of being a quadratic expression.

Example

A community center is building a pool with part of it indoors and part of it outdoors. A diagram of the indoor part of the pool is shown. The indoor shape is made from a half-circle and a rectangle. The diagram does not show the shape of the outdoor part of the pool. The exact dimensions of the pool are unavailable, but the area A of the whole pool is given by $A = \frac{\pi x^2}{2} + x^2 + 8x^2 + \frac{\pi x^2}{4}$.



Students identify the part of the expression that represents the area of the indoor part $\left(\frac{\pi x^2}{2} + 8x^2\right)$ and also the part of the expression that represents the outdoor part $\left(x^2 + \frac{\pi x^2}{4}\right)$. Then they sketch a shape for the outdoor part of the pool, which has many possibilities.

Using Expressions and Equations

In the first example, students write expressions to represent the number of border tiles needed to surround a square pool. Some of these expressions are $4s + 4$, $4(s + 1)$, $4(s + 2) - 4$, and $2s + 2(s + 2)$. An expression represents a quantity, so there is a relationship implied by an expression. Here, each expression represents the quantity or the number of tiles N. In this situation, we can say $N =$ the expression. The expression could be any one of the expressions listed above or any other equivalent expression. The implied relationship that is represented by the equation in this example is linear. That is, the relationship between the number of tiles N and the length of the square pool s is a linear relationship.

Combining Expressions

In Investigation 2, students combine expressions to write new expressions either by adding or subtracting expressions, or by substituting an equivalent expression for a given quantity in another expression that contains the quantity.

Adding Expressions

Problem 2.1 revisits a walkathon from *Moving Straight Ahead*. It provides students an opportunity to apply the Distributive and Commutative properties to write equivalent expressions for the total amount of money raised by the students in a walkathon.

> **Example**
> Leanne, Gilberto, and Alana enter a walkathon as a team. This means that each person will walk the same distance in kilometers. The walkathon organizers offer a prize to the three-person team that raises the most money.
>
> The individual pledges for each student are as follows:
>
> - Leanne has pledges from 16 sponsors. All of her sponsors pledge \$10 regardless of how far she walks.
> - Gilberto has pledges from 7 sponsors. Each sponsor pledges \$2 for each kilometer he walks.
> - Alana has pledges from 11 sponsors. Each sponsor pledges \$5 plus \$50 for each kilometer she walks.

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Students first write three expressions to represent the total amount of money raised by each student.

$M_{\text{Leane}} = 16(10)$
$M_{\text{Gilberto}} = 7(2x)$
$M_{\text{Alana}} = 11(5 + 0.50x)$

Then they use these equations to find the total by using them to calculate the money raised by each student, or by combining (adding) the three expressions to form one expression that represents the total amount of money M_{Total} raised by all three students if each walks x kilometers.

$M_{\text{Total}} = 16(10) + 7(2x) + 11(5 + 0.50x)$

Students then find an equivalent expression for the total amount of money. They interpret the information the variables and numbers represent in the new expression and discuss the advantages or disadvantages of each expression.

Making New Expressions by Substitution

In Problem 2.2, students write one linear equation to predict the profit P for an amusement park based on the probability of rain R by substituting an expression for the number of visitors into the profit equation.

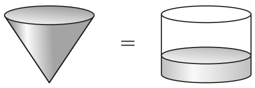
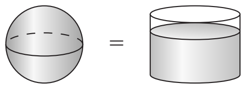
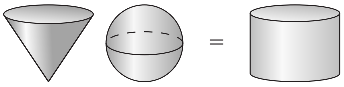
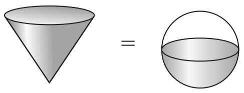
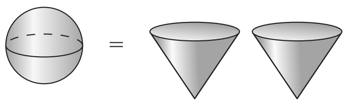
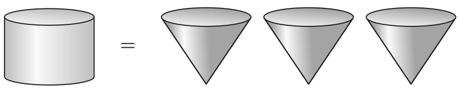
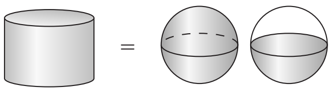
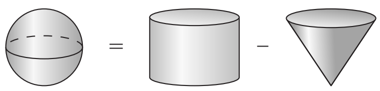
> **Example**
> The manager of the Water Town amusement park uses data collected over the past several years to write equations that will help her make predictions about the daily operations of the park. The daily concession-stand profit in dollars P depends on the number of visitors V. To model this relationship, the manager writes $P = 2.5V - 500$. To predict the number of visitors V based on the probability of rain R, she uses $V = 600 - 500R$. Write an equation that can be used to predict the profit based on the probability of rain.

This requires students to replace V in the first equation with $600 - 500R$, the equivalent expression for the number of visitors from the second equation. The equation after the substitution is $P = 2.50(600 - 500R)$. Students then write an equivalent expression for profit and compare the information each expression represents for the amusement park.

Expressions Representing Relationships Among the Volumes of a Cylinder, Cone, and Sphere

Volume of cones, cylinders, and spheres is used as a context to generate interesting algebraic expressions. Students use plastic shapes of cylinders, cones, and spheres that have the same height and radius to explore how the volumes of these shapes are related when $h = 2r$. The Pouring and Filling activity allows them to do the same thing on a computer. Before this experiment, students determine a formula for the volume of a cylinder. Some of the relationships that they discover are shown in the table:

Relationships Among the Volumes of Cylinders, Cones, and Spheres

	$V_{cone} = \frac{1}{3}\pi r^2 h$ or $\frac{2}{3}\pi r^3$
	$V_{sphere} = \frac{2}{3}\pi r^2 h$ or $\frac{4}{3}\pi r^3$
	$V_{cone} + V_{sphere} = \pi r^2 h$ or $V_{cylinder}$
	$V_{cone} = \frac{1}{2}V_{sphere}$
	$V_{sphere} = 2V_{cone}$
	$V_{cylinder} = 3V_{cone}$
	$V_{cylinder} = 1\frac{1}{2}V_{sphere}$
	$V_{sphere} = V_{cylinder} - V_{cone}$

Students then use these relationships to determine formulas for finding the volumes of the three shapes.

Solving Equations

One aspect of developing students' facility with symbols is to use equations to make predictions or answer specific questions. This sometimes requires solving equations for a specific variable.

Solving Linear Equations

Students are quite comfortable using tables or graphs to solve equations, and they can solve simple linear equations of the form $y = mx + b$, $mx + b = nx + c$, or simple equations with parentheses, such as $y = a(x + b)$. In Investigation 3, students solve more complicated equations.

> **Example**
> A school choir is selling boxes of greeting cards to raise money for a trip. The equation for the profit in dollars P in terms of the number of boxes sold s is $P = 5s - (100 + 2s)$.
>
> - What information do the expressions $5s$ and $100 + 2s$ represent?
>
> $5s$ represents the income for selling s boxes at \$5 a box. $(100 + 2s)$ represents the cost of selling s boxes; that is, \$2 per box and \$100 for miscellaneous expenses, such as advertising.
> - How many boxes must the choir sell to make a \$200 profit? Explain.
>
>   Students might use a calculator or they could substitute values for s. Some may be ready to try to solve this using the properties of equality and the Distributive Property.
> - What is the break-even point?
>
>   Students may recognize that income must be equal to cost to break even. They then set $5s = 100 + 2s$ and solve for s. Some students may use tables or graphs.
> - Write an equivalent expression for profit. What new information does this expression represent?
> - One of the choir members wrote the following expression for profit: $5s - 2(50 + s)$. Explain whether this expression is equivalent to the original expression for profit.
> - Describe how to solve an equation that has parentheses without using a table or graph.

The first few questions are similar to those that have been asked early in this Unit or in previous Units, but this equation involves more work with the use of parentheses. Students use the Distributive Property to show that the two expressions for profit, $5s - (100 + 2s)$ and $5s - 2(50 + s)$, are equivalent. In this

Problem, the focus is on developing techniques for solving equations symbolically without using tables and graphs. The last question pushes students to think about solving linear equations, the properties of equality, and the numbers that solve this equation. Next, they apply these strategies to equations like $y = 5 + 2(3 + 4x)$ or $y = 5 - 2(3 - 4x)$.

Conditional Equations, Identities, and Contradictions

The basic property of equality states that performing the same operation to each side of an equation does not change its solution. That is, if the original equality is true, then applying the properties of equality will not affect the solution.

For each example, find a value of x that makes $y_1 = y_2$. **Note:** All graphs use the window setting shown below.

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WINDOW
  Xmin=−20
  Xmax=20
  Xscl=2
  Ymin=−120
  Ymax=120
  Yscl=10
```

A *conditional equation* has at least one solution. For example, a linear equation has one solution.

Example

$y_1 = 3(2x - 5)$

$y_2 = 2(3x - 1) + x$

$$3(2x - 5) = 2(3x - 1) + x$$
$$6x - 15 = 6x - 2 + x$$
$$6x - 15 = 7x - 2$$
$$-13 = x$$

The graph of the equations is of a pair of intersecting lines, which have exactly one point in common.

```
 Plot1  Plot2  Plot3
\Y1=3(2X−5)
\Y2=2(3X−1)+X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
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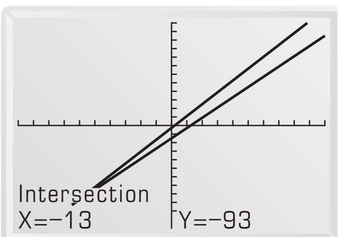

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An *identity* has an infinite number of solutions. Its solutions are true for all values of the variable.

Example

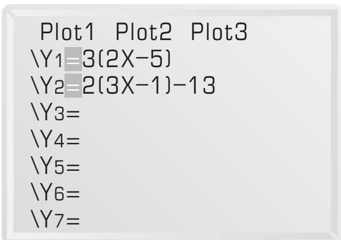
$y_1 = 3(2x - 5)$

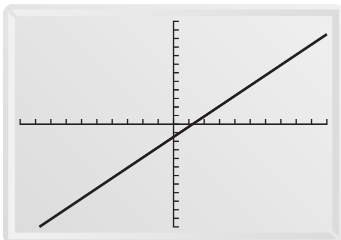
$y_2 = 2(3x - 1) - 13$

$$3(2x - 5) = 2(3x - 1) - 13$$
$$6x - 15 = 6x - 2 - 13$$
$$6x - 15 = 6x - 15$$
$$0 = 0$$ Always true.

This is an identity. All real numbers are a solution to this equation.

The graph of the equations is of a pair of equivalent lines, which have an infinite number of points in common. That is, their points all lie on the same line.





A *contradiction* has no solutions. This implies a contradiction of some known fact.

Example

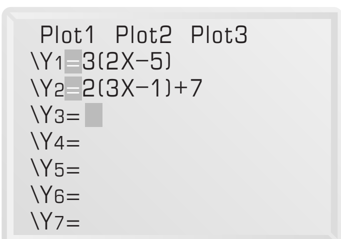
$y_1 = 3(2x - 5)$

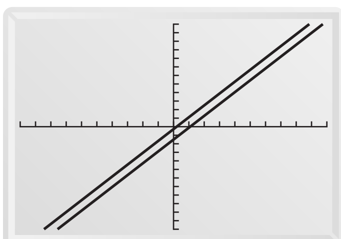
$y_2 = 2(3x - 1) + 7$

$$3(2x - 5) = 2(3x - 1) + 7$$
$$6x - 15 = 6x - 2 + 7$$
$$6x - 15 = 6x + 5$$
$$-15 = 5$$ False.

This is a contradiction. There is no solution.

The graph of the equations is of a pair of parallel lines, which have no points in common.





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In summary, an equation is called

- a conditional equation if it has at least one solution. (A linear equation has one solution.)
- an identity if it has all the numbers in a specified set as solutions.
- a contradiction if it has no solutions. It implies a contradiction of some known fact.

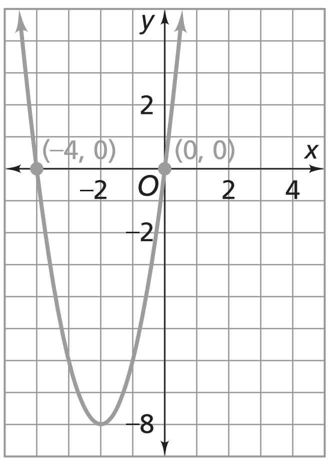
Note: In the student book, we do not use the terms, conditional, identity, and contradiction because the definitions include information about conditional equations that students have not yet encountered. For example, the equation $\sin(x) = 0.5$ is conditional, but it has an infinite number of solutions. Instead we use the language from CCSS, which is to state whether the linear equation has one solution, no solution, or an infinite number of solutions.

Solving Quadratic Equations

To solve quadratic equations like $0 = 2x^2 + 8$ or $0 = x^2 + 5x + 6$, students recognize that these equations are specific cases of the equations, $y = 2x^2 + 8$ or $y = x^2 + 5x + 6$. Finding x when $y = 0$ is the same as finding the x-intercepts of the graphs of these equations. Students have already had experience solving quadratic equations in *Frogs, Fleas, and Painted Cubes* using tables and graphs. In Investigation 3, the connection is made between solving quadratic equations for x when $y = 0$ and finding x-intercepts. Students are introduced to solving quadratic equations by factoring and solve for x when $y = 0$. They solve equations of the form $y = ax(x + b)$, $y = x^2 + bx$, or $y = ax^2 + bx + c$ that are easily factored into the product of two binomials.

Example

If $y = 2x^2 + 8x$ find the values of x when $y = 0$.



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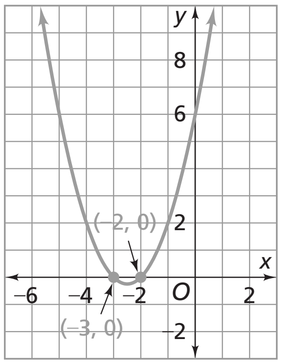
Students need to recognize that the expression $2x^2 + 8x$ can be rewritten in the equivalent form of $2x(x + 4)$. Next, they must recognize that this product can only be zero if one of the factors, $2x$ or $x + 4$, is equal to zero. This is known as the Zero Product Rule, if $ab = 0$, then $a = 0$ or $b = 0$. Thus, $2x = 0$ or $x + 4 = 0$. Solving each of these linear equations gives $x = 0$ or $x = -4$.

It is important that students understand that finding x when $y = 0$ is the same as finding the x-intercepts of the graph of $y = 2x^2 + 8x$. Important steps are factoring the quadratic expression and applying the fact that for any two real numbers, a and b, if $ab = 0$, then either $a = 0$ or $b = 0$.

Example

If $y = x^2 + 5x + 6$, find the values of x when $y = 0$.



Students write $y = x^2 + 5x + 6$ in factored form $(x + 2)(x + 3)$ and then solve $0 = (x + 2)(x + 3)$. Thus, $x + 2 = 0$, which means $x = -2$, or $x + 3 = 0$, which means $x = -3$.

Students are asked to check their solutions when solving equations. Frequently, they are also asked to connect the solutions to a quadratic equation to its graph.

Factoring

Before quadratic equations are solved, students spend time factoring quadratic expressions in Problem 3.3. It is important to note that factoring is mostly trial and error using clues from the coefficients of x, x^2, and the constant term. Given any three real numbers for a, b, and c in $ax^2 + bx + c$, it is unlikely that the expression is easily factorable, even if a, b, and c are whole numbers between 1 and 10. Over a number of years, mathematicians developed the quadratic formula that can be applied to any quadratic equation. It states that if $0 = ax^2 + bx + c$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. What is important is that students understand that quadratic expressions can be written in two equivalent forms, expanded form and a factored form, and that these two expressions represent different pieces of information about the underlying quadratic function or the context that it models. Students should be able to factor simple quadratic expressions and understand how factoring quadratic expressions uses the Distributive Property. The quadratic formula is developed in the Grade 8 Unit *Function Junction*.

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Predicting the Underlying Patterns of Change

Prior to this Unit, the focus of the algebra strand has been the study of patterns of change between two variables. Students represented these relationships using tables, graphs, and symbolic statements. They used these representations to study the special patterns of change associated with linear, exponential, and quadratic functions and the used them to solve equations. For linear situations, the statements were in the form of $y = mx + b$; for exponential situations, the statements were in the form $y = a(b^x)$; and for quadratic situations, the statements were in the form $y = ax^2 + bx + c$ or $y = (x + p)(x + q)$. Students used contexts or representations to determine whether a situation is linear, exponential, quadratic, or none of these and to write an equation to model the situation. The following discussion reviews patterns of change associated with linear, exponential, and quadratic functions.

Patterns of Change

Example
$y = 70 - 5x$

x	0	1	2	3	4	5
y	70	65	60	55	50	45
first differences	−5	−5	−5	−5	−5	

The coefficient of x, −5, is the constant rate of change and the slope for this linear relationship. The constant term, 70, is the y-intercept. When the equation represents a "real" situation, then parts of the equation are directly related to specific values in the situation. For example, a club has $70 to spend on a trip. If the trip costs $5 per person, then y or $y = 70 - 5x$ represents the amount of money left if x people go on the trip.

Example
$y = 3(2)^x$

x	0	1	2	3	4	5
y	3	6	12	24	48	96
constant factors	× 2	× 2	× 2	× 2	× 2	

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The base 2 is the constant growth factor and 3 is the y-intercept. As x increases by 1, y changes by a factor of 2. When the equation represents a "real" situation, then parts of the equation are directly related to specific values in the situation. For example, suppose the Queen of Montarek offers a peasant three rubas at the start and then doubles the amount of money each day. Then the equation $y = 3(2)^x$ represents the amount of money y on day x.

Example

$y = x(x + 1)$ or $y = x^2 + x$

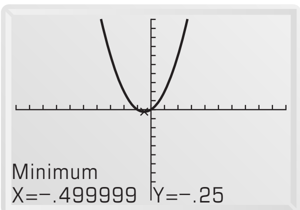
x	0	1	2	3	4	5
y	0	2	6	12	20	30

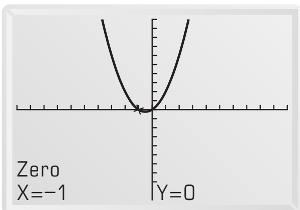
first differences: 2, 4, 6, 8, 10

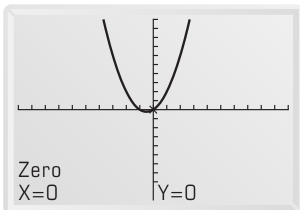
second differences: 2, 2, 2, 2

Students can recognize quadratic equations by looking at second differences of successive values of y. Writing the associated quadratic equation is a bit more challenging. In quadratic situations, the y-value grows in some relation to the square of the x-value. In this example, the y-value grows as the square of the x-value, x^2 plus x. For large values of x, x^2 is much larger than x. The equation is characterized by a constant second difference, 2. In calculus, they will learn that this difference is the second derivative of the function.

Students also recognize that the graph of $y = x(x + 1)$ has a minimum point at $\left(-\frac{1}{2}, -\frac{1}{4}\right)$, and x-intercepts are (0, 0) and (−1, 0).







This equation could represent the number of handshakes that take place between two teams, one with x members and one with $x + 1$ members. For quadratic equations, the expression for y can be written in expanded or factored form. The form to use depends on the information that is needed. To predict the y-intercept and patterns of change, the expression $x^2 + x$ is best to use. To predict the x-intercepts, line of symmetry, and the maximum or minimum points, the expression $x(x + 1)$ is best to use.

Even though the first three Investigations of this Unit focus on equivalent expressions and solving equations, students are frequently asked to describe the patterns of change that a situation or equation represent. Predicting patterns of change and interpreting the special features of a function is the focus of Investigation 4. This Investigation provides some contexts that involve more complex equations, including some that involve all three functions. Finally, Investigation 4 serves as a cumulative review for the algebra strand up to this point.

Predicting Linear Patterns of Change

In *Moving Straight Ahead*, students learned about linear relationships. In Investigation 4, students focus on a situation in which the pattern of change is linear, that is, the constant difference is the slope.

Example

Magnolia Middle School needs to empty their pool for resealing. Ms. Theodora's math class decides to collect data on the amount of water in the pool and the time it takes to empty it. They write an equation to represent the amount of water w (in gallons) in the pool after t hours.

$w = -250(t-5)$

- How many gallons of water are pumped out each hour?
- How long will it take to empty the pool?
- How many gallons of water are in the pool at the start?
- Write an expression for the amount of water in the tank after t hours that is equivalent to the original expression.
- What information does this new expression tell you about the amount of water in the tank?
- Which expression is more useful in this situation? Explain
- Without graphing the equation, describe the shape of the graph.

This linear situation contains parentheses that students have briefly encountered in *Moving Straight Ahead* and *Thinking With Mathematical Models*. To find the rate at which water is being pumped out per hour, students may use a variety of strategies.

- They may make a table and note that as t increases by 1, the water decreases by 250 gallons.
- They may substitute values into the equations and note the difference between two consecutive hours.
- They may apply the Distributive Property and write $w = -250t + 1{,}250$ and then recognize that the coefficient of t is the constant rate of change for a linear relationship.

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The amount of water at the start of the pumping is the w-intercept, which students may read from an equivalent expression for the amount of water in the tank, or they can use a table or graph. Similarly, to find how long it will take to empty the pool, they may solve the equation for $w = 0$, or use a table or graph. Some students may note that the expression is in factored form and if the amount of water w is 0, then one of the factors must be 0. So, $t - 5 = 0$ or $t = 5$. Students also describe the graph of the relationship without making a table or graph.

These questions are similar to questions asked in previous Units except that the expression for the amount of water contains parentheses. This is an example of interpreting symbolic statements to find specific information about the situation and describing the underlying relationship that the equation represents. This Problem is followed by a Problem in which students make a table of values for linear, exponential, and quadratic equations, given the same two points for each relationship.

Writing Equations for Linear, Exponential, and Quadratic Functions Given Two Points

In prior Units, students learned about linear, exponential, and quadratic relationships. In Problem 4.3, students focus on constructing patterns with a specific type of relationship from two coordinates.

Example

The first two rows in a table of numbers are given below. Write four more numbers in each column to make a linear relationship, an exponential relationship, and a quadratic relationship.

Data Points

x	Linear y	Exponential y	Quadratic y
1	1	1	1
2	4	4	4
3	■	■	■
4	■	■	■
5	■	■	■
6	■	■	■

- Explain why the relationship in each column works.
- Write an equation for each relationship. Explain what information the variables and numbers represent.
- Compare your equations with your classmates' equations. Do you all have the same equations? Explain.

For the linear and exponential relationship, each relationship has only one pattern. For the quadratic relationship, there are infinitely many patterns. This Problem shows students' understanding of the underlying pattern of change for each function.

Finally, this Investigation ends with a Problem in which students match descriptions of situations with the appropriate function type. Then they write an equation to represent each situation and describe the shape of each graph with as much detail as possible, including the pattern of change, the x- and y-intercepts, the maximum or minimum points, and the line of symmetry.

Reasoning With Symbols

In Investigation 5, the central idea is to use symbolic statements and appropriate mathematical properties to confirm conjectures. In *Prime Time*, students conjectured that the sum of two odd numbers and the sum of two even numbers are even. They tried many examples that confirmed their conjecture, and they used geometric arrangements of square tiles to validate their conjecture. They arranged rectangular arrays whose dimensions were 2 and n, where n is a whole number, to represent even numbers. The arrays for odd numbers were the same as those for even except that they had one extra piece added to the rectangle.

Some students argued that even numbers have a factor of 2, so the sum of two even numbers will have a factor of 2. For odd numbers, they argued that you are combining the two extra 1's to end up with an even number. Using symbolic statements is a way to confirm these intuitive arguments.

Using Symbolic Statements to Confirm a Conjecture

If n is any integer, then $2n$ represents an even number and $2n + 1$ represents an odd number. The sum of two even numbers is even:

Let $2m$ and $2n$ represent any two even numbers.

$$2m + 2n = 2(m + n)$$

The number $2(m + n)$ is even, so the sum of two even numbers is even.

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The sum of two odd numbers is even:

Let $2m + 1$ and $2n + 1$ represent any two odd numbers.

$$\begin{aligned}(2m + 1) + (2n + 1) &= 2m + 2n + 2 \\ &= 2(n + m + 1)\end{aligned}$$

The number $2(n + m + 1)$ is even, so the sum of two odd numbers is even.

Similar arguments can be used for the following:

- Sum of an odd and an even number
- Product of an odd and an even number

The symbolic arguments offer a very precise and convincing argument for all integers. In the next example, students look for patterns in a quadratic relationship, and then find a way to confirm their conjectures about the patterns.

Example

Perform the following operations on the first eight odd numbers. Record your information in a table.

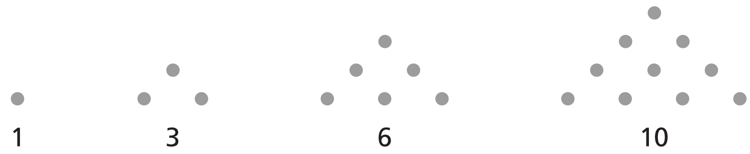
- Pick an odd number.
- Square it.
- Subtract 1.

- What patterns do you see in the resulting numbers?
- Make conjectures about these numbers. Explain why your conjectures are true for any odd number.

If students try this procedure for the first few odd numbers, they quickly see that the numbers are multiples of 8. If they rewrite each number as a product of 8, they also see that each of these numbers is 8 times a triangular number.

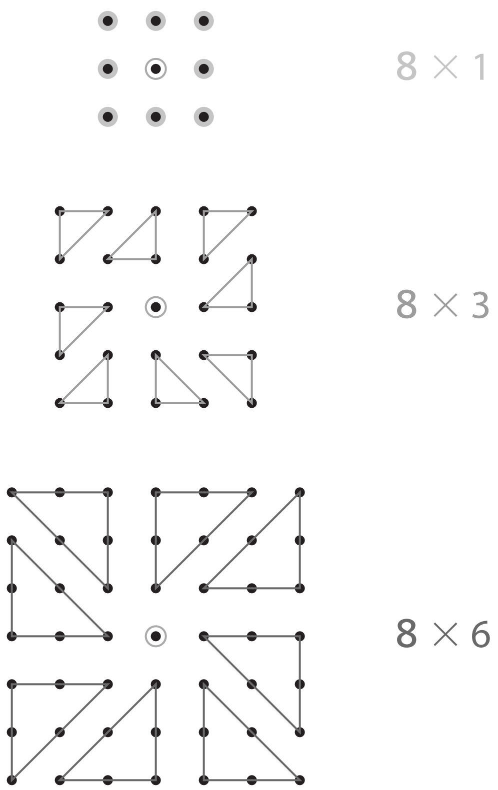
s	1	3	5	7
$s^2 - 1$	0	8	24	48
Pattern	8×0	8×1	8×3	8×6

The nth triangular number is represented by $\frac{n(n+1)}{2}$.



So, $s^2 - 1 = \frac{8n(n+1)}{2}$, where s is an odd number and $n = 1, 2, 3, \ldots$

$s^2 - 1$



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The inductive proof can be proven deductively in the following algebra proof.

Let $2n + 1$ represent an odd number.

$$
\begin{aligned}
(2n + 1)^2 - 1 &= 4n^2 + 4n + 1 - 1 \\
&= 4n^2 + 4n \\
&= 4n(n + 1) \\
&= \frac{8n(n + 1)}{2}
\end{aligned}
$$

Since $\frac{n(n + 1)}{2}$ is the nth triangular number, squaring an odd number and subtracting 1 is 8 times a triangular number.

The first part of this Problem, which involves observing patterns and making conjectures, is accessible to all students. Whether you want to help students develop a symbolic argument at this time is a decision you can make based on your students' needs. In this Unit, students

- Write symbolic expressions to represent the dependent variable in a situation
- Write equivalent expressions to reveal new information about a situation
- Interpret expressions
- Use expressions and equations to make decisions
- Solve equations and predict patterns of change that are represented by symbolic statements
- Use symbolic statements and properties of numbers to provide arguments for conjectures

Much of the work involved in this Unit could be thought of as developing traditional algebra skills, but an important difference is that students are learning these skills in a purposeful way. Not only are they using properties of numbers to write equivalent expressions, they are using expressions and equations to make important decisions about a problem situation or a function.