

▼ Mathematics Background

Systems of Linear Equations

Solving systems of equations and inequalities relies on the graphing and symbolic methods developed in this Unit. This Unit also provides students with the foundation to understand how the process of solving systems of linear equations and inequalities can be automated with computers. Representing systems as matrices provides the key to solving systems with computers. Since matrices are used extensively for linear programming and linear regression, these areas of applied mathematics were largely developed and adapted for business in the mid-twentieth century when computers became prevalent.

One of the standard topics in elementary algebra is the variety of techniques available for solving systems of two linear equations in two unknowns. In *Moving Straight Ahead*, students used the simplest form of a linear equation and were asked to find the intersection point of two lines.

Example

$$y = 3x + 5 \text{ and } y = 2x - 1$$

Students found the solution directly by realizing that the condition implies that $3x + 5 = 2x - 1$. They found the solution $x = -6$ with methods that are applicable to linear equations in one unknown. Then they substituted -6 for x in one of the original equations and found that $y = -13$.

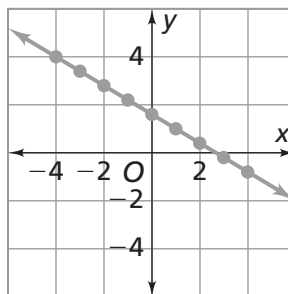
Many problems that involve conditions on two variables are represented by pairs of equations in standard form, $Ax + By = C$. The pair or set of equations is

displayed with a bracket, $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$

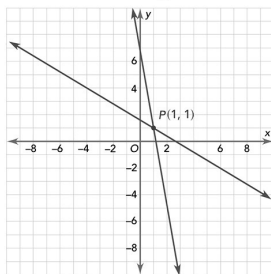
To work effectively with systems, several key ideas are important to keep in mind. First, if you plot all points (x, y) that satisfy an equation such as $3x + 5y = 8$ that is written in standard form, $Ax + By = C$, the graph is a straight line.

Example
 $3x + 5y = 8$

x	y
-4	4.0
-3	3.4
-2	2.8
-1	2.2
0	1.6
1	1.0
2	0.4
3	-0.2
4	-0.8

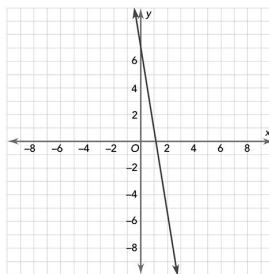


Unique Solution
Intersecting Lines



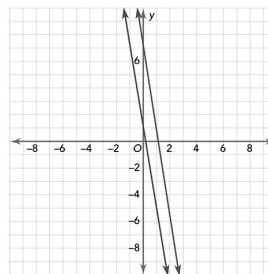
$$\begin{cases} 6x + y = 7 \\ 3x + 5y = 8 \end{cases}$$

Infinite Solution Set
Coincident Lines



$$\begin{cases} 6x + y = 7 \\ 12x + 2y = 14 \end{cases}$$

No Solution
Parallel Lines



$$\begin{cases} 6x + y = 7 \\ 6x + y = 1 \end{cases}$$



Methods for Solving Systems of Equations

The standard solution strategies for systems of equations are important for students later in mathematics, so a conceptual understanding is a useful foundation that students can build on.

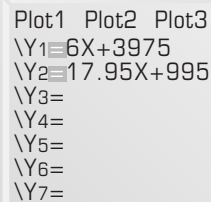
Graphing

The graphic method involves graphing all the equations in a system on a single coordinate grid in order to read the coordinates of the intersection point(s) as the solution(s). Since this method relies on a visual representation of the equations, coordinates of an intersection point can only be estimated and may not even appear in the graphing window you select to display. If a display does not show any intersection point(s), you may still have to resort to hand calculations to determine a graphing window that shows the solution(s). Since the coordinates of an intersection point are estimates, it is important to check the solution(s) in the original equations. When you suspect that lines are parallel, you will want to find the slopes of each line to see if their slopes are equal.

In Problem 3.1, Question A, students use graphs of $y_1 = 6x + 3,975$ and $y_2 = 17.95x + 995$ to estimate the solution of the system. If you are using graphing calculators in class, then finding the correct window may be challenging.

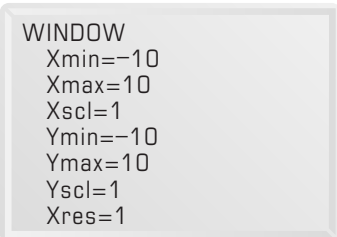
Graphing Calculator Example

Enter the functions into the function window.

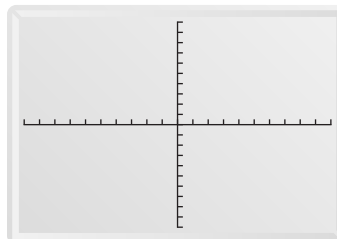


```
Plot1 Plot2 Plot3
\Y1=6X+3975
\Y2=17.95X+995
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

Now, if you select **GRAPH** using the standard window, your display will not display any lines because they are outside the window.

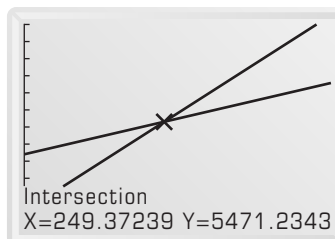


```
WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-10
Ymax=10
Yscl=1
Xres=1
```



Change your window settings as shown below and select **GRAPH** to see the lines. Find the point of intersection by pressing **CALC** and selecting the intersect option on the menu.

WINDOW
 Xmin=0
 Xmax=500
 Xscl=50
 Ymin=0
 Ymax=10000
 Yscl=1000
 Xres=

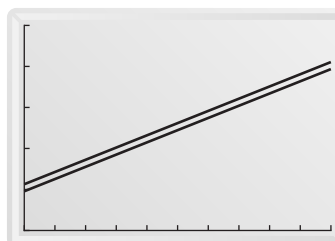


Graphing Calculator Example

To demonstrate why you may have to check slopes when you are trying to find an intersection point graphically, suppose that the equations are $y_1 = 6x + 3,975$ and $y_2 = 5.95x + 4,025$ instead of $y_1 = 6x + 3,975$ and $y_2 = 17.95x + 995$.

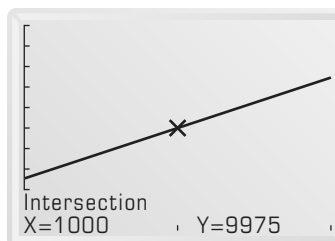
If you use the window settings shown below, the lines appear parallel.

WINDOW
 Xmin=0
 Xmax=100
 Xscl=10
 Ymin=3800
 Ymax=4800
 Yscl=200
 Xres=1



If you change the window settings using Zoom In on the Zoom menu, you might get the display of the window and graph below.

WINDOW
 Xmin=800
 Xmax=1200
 Xscl=100
 Ymin=7500
 Ymax=12500
 Yscl=500
 Xres=1



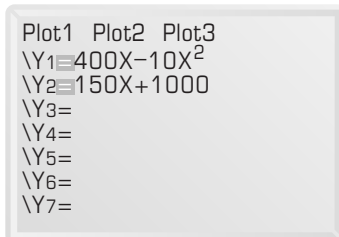
This example shows that comparing slopes to see if they are equal is a solid first step even when using the graphing method.

continued on next page

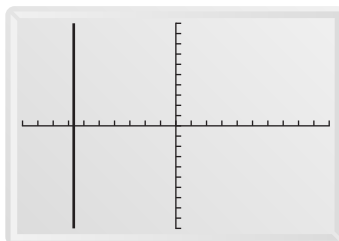
Graphing Calculator Example

In Problem 3.3, Question A, students use graphs of $y_1 = 400x - 10x^2$ and $y_2 = 150x + 1,000$ to estimate the solution of the system. If you are using graphing calculators in class, then finding the correct window may be challenging, again.

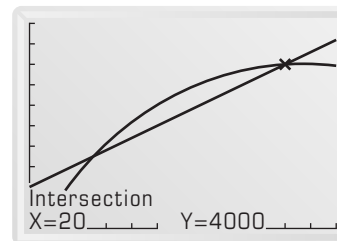
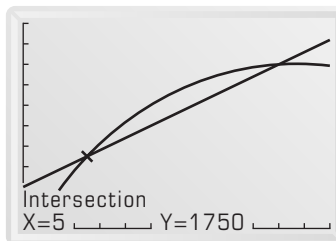
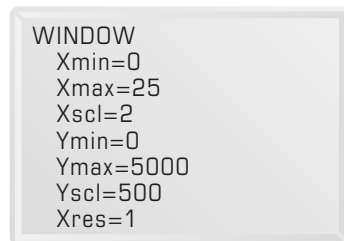
Enter the functions into the function window.



Note: The standard window settings produce the graph below.



To see the intersection points on the display, select the window below and graph. Then using the **CALC** feature, you can display the intersection points one at a time.



Equivalent Form

When the equations in a system are given in standard form, $Ax + By = C$, they can be rewritten in $y = ax + b$ form. When the arithmetic is easy, this is a good strategy.

Example

$$\begin{cases} 2x + y = 5 \\ 9x - 3y = 15 \end{cases}$$

This system is equivalent to $\begin{cases} y = -2x + 5 \\ y = 3x - 5 \end{cases}$.

Graphing this system is now easy. Also, setting the two expressions for y equal to each other eliminates a variable. That means you can solve $-2x + 5 = 3x - 5$ for x using methods for solving linear equations

$$\begin{aligned} -2x + 5 &= 3x - 5 \\ -2x + 5 + 2x &= 3x - 5 + 2x \\ 5 &= 5x - 5 \\ 5 + 5 &= 5x - 5 + 5 \\ 10 &= 5x \\ 2 &= x \end{aligned}$$

Since $x = 2$, $3x + 5 = 3(2) - 5 = 1$, or $y = 1$.

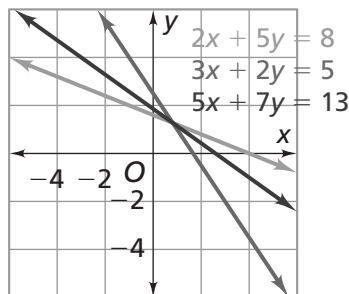
Linear Combination

A third method of solving linear systems relies on two basic principles:

1. The solutions to any linear equation $ax + by = c$ are identical to the solutions of the equation $kax + kby = kc$, where $k \neq 0$. That is, multiplying each side of a linear equation by the same nonzero number does not change the set of solutions, or $kax + kby = kc$, is equivalent to $ax + by = c$.
2. The solution to any system of linear equations is unchanged if one of the equations is replaced by a new equation formed by adding the two original equations.

Example

$\begin{cases} 2x + 5y = 8 \\ 3x + 2y = 5 \end{cases}$ has exactly the same solution as the system $\begin{cases} 2x + 5y = 8 \\ 5x + 7y = 13 \end{cases}$.



continued on next page

This property is often explained by invoking a general “equals added to equals are equal” rule, though careful justification requires more substantial reasoning. Using the two properties of the linear combination method, one of the two variables in a system can be eliminated. That is why this method is commonly referred to as the elimination method.

Example

$$\begin{cases} 3x + 5y = 8 \\ 6x + y = 7 \end{cases}$$

By property (1) of the linear combination method, $\begin{cases} 3x + 5y = 8 \\ 6x + y = 7 \end{cases}$ is equivalent

to $\begin{cases} -6x - 10y = -16 \\ 6x + y = 7 \end{cases}$ using $k = -2$ with the first equation. By property (2),

$\begin{cases} -6x - 10y = -16 \\ 6x + y = 7 \end{cases}$ is equivalent to $\begin{cases} -9y = -9 \\ 6x + y = 7 \end{cases}$. Solving the first equation for y

shows that $y = 1$. Substituting $y = 1$ into the second equation gives $x = 1$.

As a general rule, use of the linear combination method requires some numerical intuition in order to select and use an efficient choice of combination operations. There are systematic algorithms that remove the intuition process; those algorithms are available on most graphing calculators and computer mathematics software. Often, the systematic linear combination procedures involve complicated calculations with fractions, which can trip all but the very best human calculators! The important goal for treatment of linear combinations in this Unit is to lay a foundation of intuition about the key ideas involved, not to develop high levels of proficiency in implementing the strategy.

Types of Solutions

When solving a system of two linear equations symbolically using the linear combination method, students will encounter cases where there is no solution or where there are infinitely many solutions.

Example

$\begin{cases} 4x + 2y = 10 \\ 2x + y = 4 \end{cases}$. Use property (1) with $k = -2$ to transform the second equation

to the system $\begin{cases} 4x + 2y = 10 \\ -4x - 2y = -8 \end{cases}$. Use property (2) to combine the equations

resulting in $0 = 2$, which is a contradiction.

Since $0 = 2$ is false, there is no solution to this system; the solution set is the empty set. That is the two equations must not have a common solution because the assumption that there are values for x and y that satisfy both equations has led to a contradiction. Connecting to the graphic method for solving this system, this solution makes sense because both equations have the slope -2 , which verifies that their graphs are parallel lines and do not intersect.

Example

$$\begin{cases} 4x + 2y = 10 \\ 2x + y = 5 \end{cases}$$
 Use property (1) with $k = -2$ to transform the second equation to the system

$$\begin{cases} 4x + 2y = 10 \\ -4x - 2y = -10 \end{cases}$$
 Use property (2) to combine the equations resulting in $0 = 0$.

Since $0 = 0$ is true regardless of the values of x and y , the solution to this system is any pair of values for x and y that satisfies either equation. That is, the system of equations has infinitely many solutions. Connecting to the graphic method for solving this system, this solution makes sense because both equations simplify to the same equation $y = -2x + 5$; they both represent the same line and all the same points. The set of points that satisfies the pair of linear equations is infinite.

Linear Inequalities

Connected Mathematics students spend a considerable amount of time working with coordinate graphs of linear equations in the form $y = mx + b$. These graphs are connected to the solution of linear equations such as $5x + 7 = 42$. Your students have not spent as much time solving linear inequalities such as $5x + 7 \leq 42$ and interpreting the solutions using number lines or coordinate graphs.

There are three basic ways to look at inequality problems: symbolically, graphically, and numerically. The first two ways are primarily used to solve inequalities mathematically.

Example

$$5x + 7 \leq 42$$

To solve the inequality symbolically, apply the same operations (addition, subtraction, division, and multiplication) that are used to solve the related linear equation to each side of the inequality, except for multiplication and division by a negative number.

Solving the Inequality

$$5x + 7 \leq 42$$

$$5x \leq 35 \quad \text{Subtract 7 from each side.}$$

$$x \leq 7 \quad \text{Divide each side by 5.}$$

Solving the Equality

$$5x + 7 = 42$$

$$5x = 35$$

$$x = 7$$

When solving linear inequalities, students learn the special properties that govern rewriting inequalities into equivalent and simpler forms. The rules for operations with inequalities are almost identical to those for operations with equations. The exception is that multiplying or dividing each side of an inequality by a negative number reverses that the direction of the inequality symbol.

continued on next page

Example

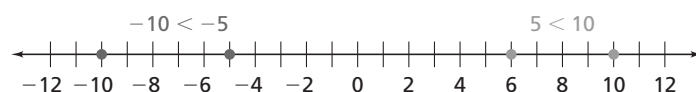
$$-5x + 7 \leq 42$$

$$-5x + 7 \leq 42$$

$$-5x \leq 35 \quad \text{Subtract 7 from each side.}$$

$$x \geq -7 \quad \text{Divide each side by } -5. \text{ Reverse the inequality's direction.}$$

To see why the direction of the inequality sign reverses, think of the number line with positive and negative numbers as mirror images of each other across zero. Thus, 5 is to the left of 10, but -5 is to the right of -10 .



Multiplying or dividing by a negative value will change positive quantities to negative quantities, or vice versa. The order of the quantities will be reversed, so that if $5 < 10$, dividing by -1 on each side of the inequality results in $-10 < -5$.

Another approach is to start by adding $5x$ to each side of the inequality to avoid the final step of reversing the sign.

$$-5x + 7 \leq 42$$

$$7 \leq 42 + 5x \quad \text{Add } 5x \text{ to each side.}$$

$$-35 \leq 5x \quad \text{Subtract 7 from each side.}$$

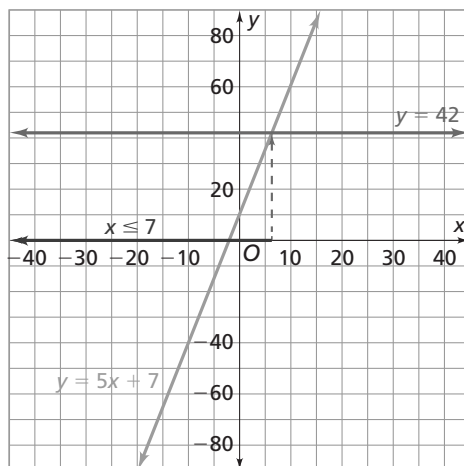
$$-7 \leq x \quad \text{Divide each side by 5.}$$

The second way to approach solving an inequality is graphically. You can graph each side of the inequality as a separate function.

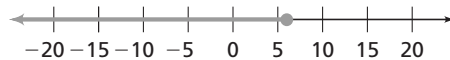
Example

$$5x + 7 \leq 42$$

Graph $y_1 = 5x + 7$ and $y_2 = 42$.



Note that the intersection of these two lines satisfies the equation $5x + 7 = 42$ at $x = 7$. However, to find when $5x + 7 \leq 42$ requires finding the points on the line $y = 5x + 7$ satisfying the condition that the y -coordinate is less than 42. The x -values that are less than or equal to 7 are the points on the graph of the line where $5x + 7 \leq 42$. These points are shown on the graph as the bold part of the line along the x -axis. When you examine the graph ask, what values of x when substituted into $y = 5x + 7$ give a value of $y \leq 42$? This strategy is more efficient with a graphing calculator. To show the solution, the values $x \leq 7$ can be graphed on a number line.



The third way to approach solving an inequality such as $5x + 7 \leq 42$ is to examine a table of values.

Example

$$5x + 7 \leq 42$$

Plot1	Plot2	Plot3
$\backslash Y_1 = 5X + 7$		
$\backslash Y_2 =$		
$\backslash Y_3 =$		
$\backslash Y_4 =$		
$\backslash Y_5 =$		
$\backslash Y_6 =$		
$\backslash Y_7 =$		

X	Y ₁
3	22
4	27
5	32
6	37
7	42
8	47
9	52

$Y_1 = 42$

Then examine the table to determine the values of x for which the y -value is less than or equal to 42. As you can see from the table, the solution is $x \leq 7$. This strategy is also more efficient with a graphing calculator.

Solving Linear Inequalities in Two Variables

In practical problem situations where linear conditions constrain the variables, realistic models of the problem conditions called *constraints* are often expressed as linear inequalities, $ax + by < c$, $ax + by > c$, $ax + by \leq c$ or $ax + by \geq c$, rather than equations. With inequalities, the set of solutions (x, y) corresponds to the points in half of the coordinate plane, rather than points on a single line. The only practical way to represent the solution set is to graph it on a coordinate grid by shading the half-plane of interest. This is a standard convention.

The basic strategy that works well for graphing solutions of linear inequalities is to first graph the boundary line and then check a point from either side of that boundary line to see if it satisfies the inequality condition. If it does, then shade the side of the boundary line that it is on. If it doesn't, then shade the other side of the boundary line. In many cases, the easiest point to check is $(0, 0)$.

Solving Systems of Linear Inequalities by Graphing

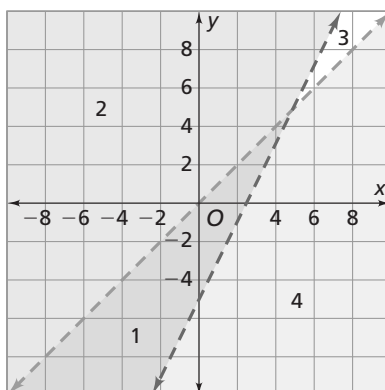
Systems of inequalities tend to have infinite solution sets and not a unique solution. The solution to a system of linear equations is the intersection of two lines. If the lines are distinct and not parallel, the intersection is a single point. The solution to a system of distinct linear inequalities is the intersection of two half-planes, which contains infinitely many points if the sets are not disjoint. If the sets are disjoint, there is no solution.

In general, there are four regions suggested by a system of linear inequalities.

Example

$$\begin{cases} y < x \\ y > 2x - 5 \end{cases}$$

A graph of the inequalities with the regions numbered 1–4 is shown below.



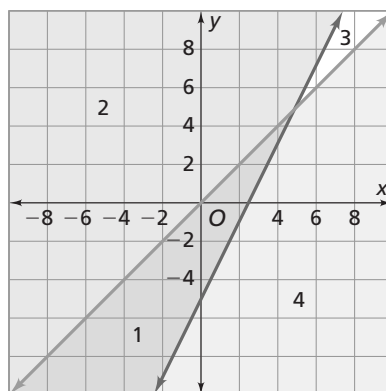
The graph shows that Region 1 contains the solutions to the system; all of the points in that region satisfy both inequalities. Regions 2 and 3 contain points that satisfy one or the other inequality but not both of them. These points are not solutions to the system, although each point is a solution to one or the other inequality. Region 4 contains points that do not satisfy either of the inequalities. These points are also not solutions to the system.

To show the boundaries for strictly less than or strictly greater than inequalities, the convention is to graph the boundary lines as dashed lines; however, if the inequality is less than or equal to or greater than or equal to, the convention is to graph the boundary lines as solid lines.

Example

$$\begin{cases} y \leq x \\ y \geq 2x - 5 \end{cases}$$

A graph of the inequalities with the regions numbered 1–4 is shown below.



The graph shows that Region 1 contains the solutions to the system. All of the points in that region and *on the borders of that region* satisfy both inequalities. Regions 2 and 3 contain points that satisfy one or the other inequality, but not both of them. These points are not solutions to the system, although each point is a solution to one or the other inequality. Region 4 contains points that do not satisfy either of the inequalities. These points are also not solutions to the system.

Systems of Lines and Curves

This Unit relates systems of linear equations to systems of inequalities. Much of the thinking used to solve linear systems is applied to systems of equations (and inequalities) with one or more nonlinear equations.

continued on next page

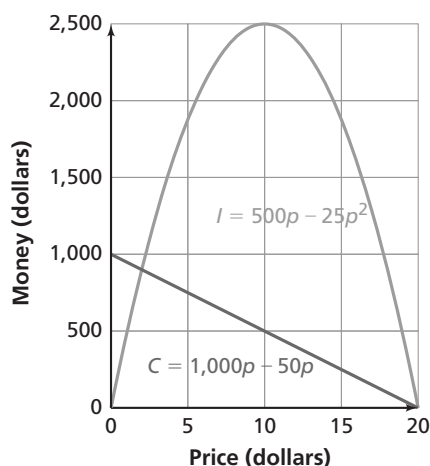
Example

A simple business plan model has a quadratic income function and a linear cost function which depend on price p . Find the break-even points, the points where income is equal to cost.

Income $I = -25p^2 + 500p$

Cost $C = 1,000 - 50p$

The income graph is a parabola, and the cost graph is a straight line. When you graph them on the same coordinate grid, the break-even points of the business plan model are the intersection points of the two graphs, as shown below. You can locate and estimate the coordinates of those points using the graph.



You can find the exact coordinates of the break-even points by solving the equation $-25p^2 + 500p = 1,000 - 50p$. This equation is equivalent to the standard form quadratic equation $-25p^2 + 550p - 1,000 = 0$ or $p^2 - 22p + 40 = 0$ with solutions $p \approx 2$ and $p \approx 20$. This business model predicts that income is equal to cost when the price is \$2 or \$20.

You can also use the graph and the algebraically derived coordinates of the intersection points to answer questions that are modeled by inequalities.

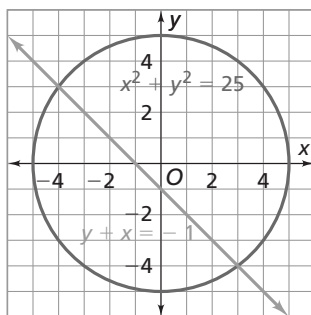
Example

Suppose you want to know when income is greater than cost based on the price p . You can find the answer by solving $-25p^2 + 500p > 1,000 - 50p$. Using the results from solving the previous system of equations, $p \approx 2$ and $p \approx 20$, and reasoning from the graphs of the two functions, you can conclude that the solution of the system of inequalities is $2 < p < 20$. That is, income will be greater than cost in this model for a price between \$2 and \$20.

Common systems involving lines and curves are encountered in coordinate geometry.

Example

Find the intersection points of the line with equation $x + y = -1$ and the circle with equation $x^2 + y^2 = 25$. You can estimate the intersection points from the graph.



You can also find the solutions by solving the system $\begin{cases} x^2 + y^2 = 25 \\ x + y = -1 \end{cases}$. To solve the system, you can first solve for y to find y^2 .

$$x + y = -1$$

$$y = -x - 1$$

$$y^2 = (-x - 1)^2$$

$$y^2 = x^2 + 2x + 1$$

Then substitute the expression for y^2 into $x^2 + y^2 = 25$, which will lead to a quadratic equation that you can solve to find x .

$$x^2 + (x^2 + 2x + 1) = 25$$

$$2x^2 + 2x + 1 = 25$$

$$2x^2 + 2x + 1 - 25 = 25 - 25$$

$$2x^2 + 2x - 24 = 0$$

$$x^2 + x - 12 = 0$$

$$(x - 3)(x + 4) = 0$$

$$x = 3 \text{ or } x = -4$$

Then substitute $x = 3$ and $x = -4$ into $y = -x - 1$. The intersection points are $(-4, 3)$ and $(3, -4)$.

Such simple examples of a system of a line and a curve are included in this Unit to address an explicit statement in the Common Core State Standards for Algebra I. However, the development does not go far, since the topic will be extended in subsequent study of advanced algebra.