The Genre(s) of Argumentation in School Mathematics

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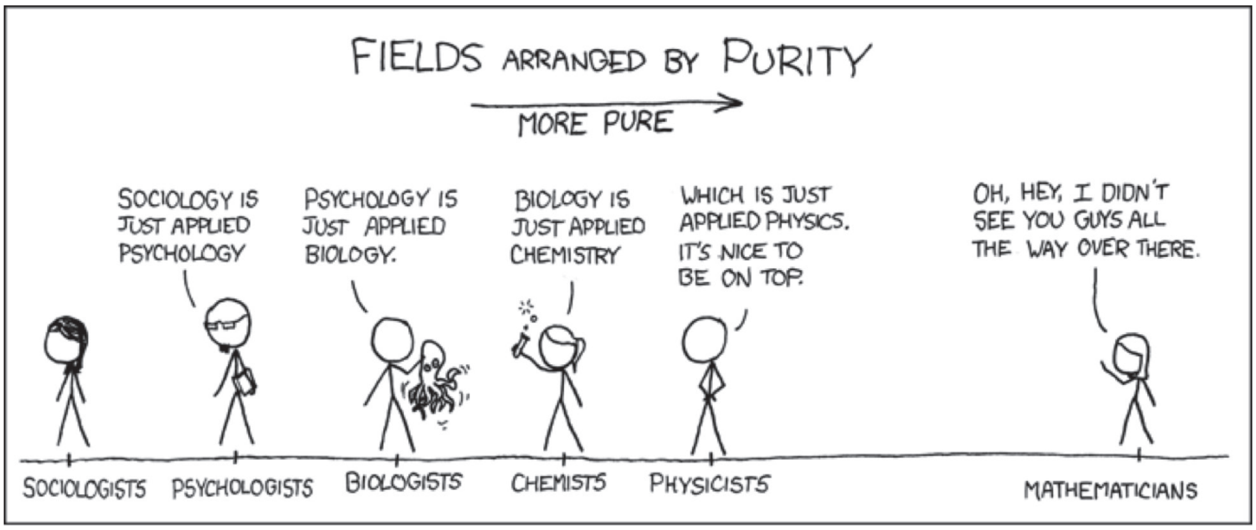


Figure 1. Purity of mathematics across fields (Munroe, 2012).

Some see mathematics as a pure and unadulterated expression of our logical and rational faculties. From this commonplace perspective, even heavily mathematical disciplines such as statistics or physics are somehow lesser chimera, beautiful reflections that are nonetheless impure. In truth, this perspective does a very real disservice to mathematics, ignoring as it does the very human and aesthetic considerations of those who practice mathematics (Ernest, 1991; Lakoff & Núñez, 2000). Mathematics, as a body of knowledge, owes much to the art of rigorous logical argument. Rigor and abstraction are manifestations of values held by the community of mathematicians that set norms for how mathematicians communicate with one another as well as how understanding is conceived within the discipline.

In this article, we seek to: (1) contribute to the breaking down of disciplinary boundaries between literary arts and mathematics by articulating the role that argument, as a literary convention, plays in mathematics, and (2) to explore how viewing mathematical argumentation as a genre of literature (or discourse more broadly) can help us see how teachers and curricula frame students' engagement in argumentation in ways that have consequences for students' understanding and social practice. In the first section of our article, we focus on exploring what it means to think about (parts of) mathematics as a genre of literary argument, and we will try to flesh out some of the characteristics that distinguish "mathematical" argument from the broader collection of genres of argument. In the second section, we explore how specific genres or subgenres of argumentation are suggested in curricular tasks and the implications of that framing for students' understanding of the role of argumentation in mathematics and what kinds of responses they are expected to produce.

Mathematics as Argumentation

In this section, we focus on characterizing "mathematical" argument as a literary genre. What does mathematical argument look like? What kinds of social values does mathematical argument uphold, and how do those values manifest in the way such arguments are expressed? What are the authors of such arguments trying to accomplish? We explore these and other questions, first with a narrative overview of the role argument plays in mathematics, then with a more careful characterization of such argument. This discussion lays the foundation for the next part of our article where we will highlight the implications of particular mathematical tasks through the lens of genre theory.

A Narrative Overview of Mathematical Argument

The barriers between mathematics and literacy are not nearly so stark as they are commonly perceived; one could even warrant arguments for claiming these barriers are altogether imaginary. Mathematics, as in the visible work of mathematicians, is largely comprised of argument. This is of key importance in the context of K–16 education precisely because the standards, assessments, and curricula of mathematics classes are all constructed using the practice of mathematicians minimally as a point-of-reference, and often as a hypothetical end goal. Mathematical proofs, constructed by mathematicians and published in journals of mathematics, are really just arguments that follow a specific set of logical rules and social norms, but they are the standard against which all mathematical practice and curricula is measured and understood. Each proof is a pyramid of claims and arguments supporting those claims carefully constructed to verify, explain, systematize, discover, communicate, or sanction (Hanna, 2000) a proposed mathematical fact. Even the less formal work of mathematicians and the broad array of practitioners of mathematics (e.g., engineers, statisticians, students) can typically be characterized as argument, albeit usually with lower bars of rigor or modified social norms. In the context of K-12 classrooms, there is often pedagogical value in foregrounding the value of understanding and communication in mathematical discussions over the value of formal rigor (sanctioning) or systematization; in other words, making space for rough draft reasoning and experimentation at the expense of strict and unabated adherence to rigid logical form and function. Mathematical discourse, written and spoken, is often intended to convince either oneself or others of the truth of some claim through appropriate justification and argumentation.

Characterizing Mathematical Argument

What makes argumentation in mathematics different from argumentation present in an essay, a piece of journalism, or a historical treatise? What makes an argument "mathematical"? It may be natural to point at mathematical symbols and Greek letters and proclaim that these are features that render an argument mathematical. Arguing this way puts at the center of a genre of writing its form and features. Genre theory, however, informs us that although textual features can be constituent parts of a specific genre, a genre is "centered not on the substance or the form of discourse but on the action it is used to accomplish" (Miller, 1984, p. 151). With this in mind, we believe that mathematical argumentation is not so different from argumentation in other disciplines. Specifically, argumentation in mathematics and English classes share more than separates them.

What action is mathematical argumentation supposed to accomplish? Standards, assessments, and curricula of mathematics are typically designed to make students more like mathematicians in the way they argue. Thus, the manner in which mathematicians view argumentation merits attention as a point of reference. In pursuit of this reference point, let us observe the significance mathematicians place upon arguments:

> By concentrating on what, and leaving out why, mathematics is reduced to an empty shell. The art is not in the "truth" but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is the art of explanation. (Lockhart, 2009, p. 5)

This quote suggests a key feature of how mathematicians view arguments (the why) in relation to their conclusions (the what): the argument that leads to a theorem is often found to be more riveting than the theorem itself. Accordingly, an argument ideally serves two main functions in mathematics: (1) as its Latin root suggests, an argument should preferably prove a result; and (2) an argument should provide the reader with some form of insight. As pointed out, it is arguably the latter purpose that excites mathematicians more. Arguments link ideas, connect what was not connected, and give rise to new ideas. An argument that proves a theorem but lacks new insights is often a disappointment for mathematicians.

How does argumentation in mathematics compare to argumentation in literacy? Consider the following standard from the *Common Core State Standards:* "Write arguments to support claims with clear reasons and relevant evidence" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 42). Is this a mathematics standard or an English language arts and literacy standard? It is, in fact, the English language arts and literacy standard CCSS.ELA-LITERACY.W.6.1.[1] The fact that this standard could be both a mathematics and an English language arts and literacy standard illustrates that the lines between argumentation in mathematics and literacy are blurry at best. More specifically, this literacy standard demonstrates that one of the purposes of a literary argument is to prove a claim, much like for a mathematical argument.

What about the second purpose of arguments in mathematics to provide the reader with insight? We believe that this is also a purpose of arguments in literacy. If we look at the distribution of communicative purposes found in the *Writing Framework for the 2011 National Assessment of Educational Progress* (National Assessment Governing Board, U.S. Department of Education, 2010), we see that as students get older and are expected to move closer to expert behavior, the expectation is that their writing shifts from the conveying of experience (the what) to persuasion and explanation (the why). As in mathematics, we see evidence that the focus on why an argument is true takes on a prominent role in literacy.

We hope that at this point you will agree that the purposes of argumentation in mathematics and literacy are nigh identical. As a matter of fact, you may be wondering, is there any difference between the genres of mathematical and literary argumentation—textual regularities (i.e., form) aside? We believe that there is only a subtle difference which relates to the form persuasion takes. As noted in the *Writing Framework for the 2011 National Assessment of Educational Progress* (National Assessment Governing Board, U.S. Department of Education, 2010), persuasion is a communicative purpose of writing that is exceedingly emphasized as students progress through school. The author of a

[1] CCSS.ELA-LITERACY.W.6.1 is identical to CCSS.ELA-LITERACY.W.7.1 and CCSS.ELA-LITERACY.W.8.1. Furthermore, CCSS.ELA-LITERACY.W.6.1 is the essence of CCSS.ELA-LITERACY.W.9-10.1 and CCSS.ELA-LITERACY.11-12.1.

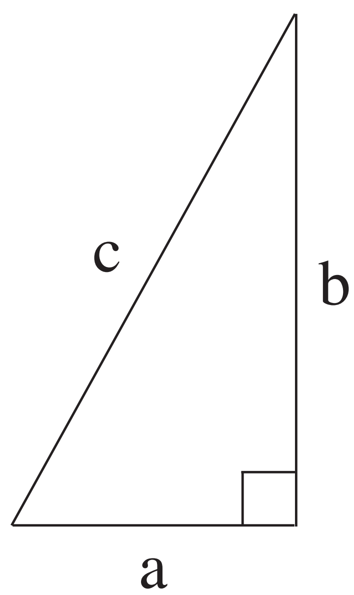
mathematical argument, however, rarely sets out to be persuasive in any colloquial sense; the logical validity of a mathematical argument supplants all necessity for traditional persuasiveness. Persuasion is essentially "outsourced" to logic. Yet, we would like to make explicit one caveat: mathematicians are not ignoring persuasion, they are simply persuaded within a different set of social values and norms. For instance, the prominence of the author or adherence to mathematical norms and conventions may persuade readers that a mathematical argument is valid even when it is not. We view this, however, not as a feature of mathematical proof but, rather, of human nature. In summary, if we consider a genre not by its textual features but by its purpose, argumentation in mathematics and argumentation in literacy are closely related genres whose only difference may be the norms of persuasion.

In this section, we have attempted to highlight both how argumentation in mathematics and literacy is substantively similar, and to describe some of how mathematical argumentation is uniquely "mathematical." In the next section, we use genre theory to critically examine implied curricular genres and subgenres with an eye for how they present opportunities for students to engage in argumentation. We then discuss how these opportunities both empower students to understand argumentation's role in doing mathematics as well as limit students' agency for authoring mathematical knowledge.

Mathematical Tasks and the Implications of Genre

We would now like to illustrate the nature of a mathematical argument in the setting of middle school mathematics by analyzing a task sequence for one of the most well-known theorems in mathematics: the Pythagorean Theorem. We present examples of students' experiences with mathematical argument not only because the Pythagorean Theorem is a well-known mathematical theorem, but because it is often the place where mathematics textbooks introduce norms of formal argumentation to students. In this section, we will present task sequences from students' study of the Pythagorean Theorem: one sequence from an early version of a textbook series and one sequence from a later version of the same textbook series. We will analyze how the presentation of tasks and the wording of prompts for student responses evokes a particular genre of mathematical argumentation and discuss the potential implications for students' understanding of mathematical argumentation. Further, we will highlight how shifts in the prompts between versions which emerged due to the publication of the Common Core State Standards for Mathematics (CCSSI, 2010) potentially changes how students engage with producing arguments to justify the Pythagorean Theorem. Our goal is to draw attention to how the sequencing of tasks and the particular words used in prompts shape students' understanding of the genre of mathematical argumentation.

The Pythagorean Theorem states that for any right triangle, the sum of the squares of the legs (i.e., the two sides that meet to form the right angle) is equal to the square of the length of the hypotenuse (Figure 2).



$$a^2 + b^2 = c^2$$

Figure 2. Representation of the Pythagorean Theorem (Baelde, 2013).

One of the possible reasons this theorem is well-known is that it is often the context for students' first encounter with proof. Many different representations of proof exist for the theorem; there are over 300 known proofs of the Pythagorean Theorem (for one example, see Figure 3).

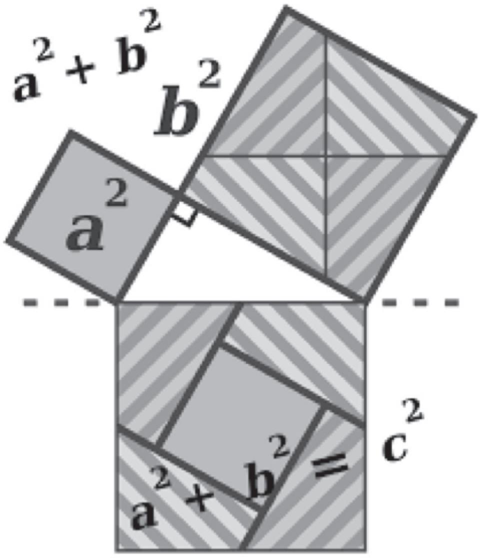


Figure 3. A picture proof of the Pythagorean Theorem (Faulk, 2014).

The task sequences we present below come from two editions of the Connected Mathematics Project (CMP 1/CMP 3) curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997; Lappan, Phillips, Fey, Friel, Grant, & Stewart, 2014). CMP 1 and CMP 3 were written to align with recommendations in the National Council for Teachers of Mathematics Principles and Standards for School Mathematics (2000) and CMP 3 incorporates recommendations from the Common Core State Standards (CCSSI, 2010). CMP is the most widely used middle school curriculum in the world and has been shown to have a relatively high percentage of problems that engage students in mathematical reasoning and argumentation (Stylianides, 2009). In both editions, students' work with the Pythagorean Theorem occurs in a series of investigations in a CMP textbook entitled *Looking for Pythagoras*.

CMP 1: Puzzling Toward a Rule

At the beginning of the *Looking for Pythagoras* unit in the first version of the curriculum (CMP 1), the CMP authors offer the following description of the learning goals for the unit: "In *Looking for Pythagoras*, you will explore an important relationship among the side lengths of a right triangle.... The unit should help you to…..understand and apply the Pythagorean Theorem." (Lappan et al., 1997, p. 4) In the first lesson, students explore and calculate distances between points placed on a dot grid (Figure 4).

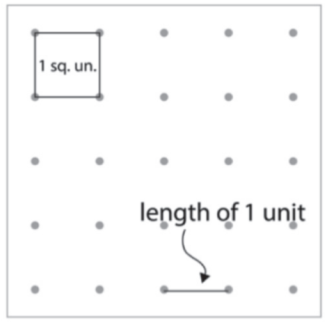


Figure 4. One square unit on dot grid (Auer, 2013).

In the next lesson, students explore how to find areas of polygons drawn on the dot grid by studying the connection between the side lengths of the shapes and their area. This exploration helps students understand the concept of a square root; to find the length of a side of a square, you would first compute the area (area = length x width) by counting the number of one square units inside the polygon drawn on the dot grid. Then, since the length and the width of the square are the same, you can take the square root of the value for the area to find the side length of the square. Students can then use the dot grid to find the length of any line segment by first constructing a square with the segment as a side length and then calculating the area of the square.

In the homework problems for this lesson, we begin to see students being invited to think about general cases and to construct arguments that are initially grounded in their own understanding and sense-making:

> "Find **every possible** area for a square drawn by connecting dots on a three-dot by three-dot grid." (Lappan et al., 1997, p. 22)

> "Find the areas of triangles AST, BST, CST, and DST. How do the areas compare? **Why do you think this is true?**" (Lappan et al., 1997, p. 23)

This lesson sequence prepares students to think about measures of all the side lengths of right triangles: the two legs and the other side opposite the right angle (called the "hypotenuse"). The dot grid makes it possible to generate measurements without using a ruler. By asking students to think about "every possible" area for a square drawn, they must grapple with conceptualizing the range of cases that would apply in this situation, asking themselves questions like: "Is this everything? What would not fit in this set?" Here we see the initial invitations to forming a generalization. Articulating generalization is not only an important part of students beginning to understand abstract mathematical ideas, but it is also a core social value of the discipline of mathematics. In creating mathematics as a body of knowledge, mathematicians primarily concern themselves with phenomena that apply to many cases such as mathematical ideas and procedures that work for all numbers. In the context of literacy, we can also think of generalization as an outcome of synthesizing information to draw a conclusion (CCSS.ELA-LITERACY. WHST.6-8.1.B) "Support claim(s) with logical reasoning and relevant, accurate data and evidence that demonstrate an understanding of the topic or text, using credible sources" (CCSSI, 2010).

Another question in this task sequence invites students to think about a pattern they notice in areas of triangles and why they think this is true. This question might have been phrased, "Show/justify/explain why the areas of the triangles are equal" but asking instead, "How do the areas compare? Why do you think this is true?" allows a range of students' ideas to surface through an ambiguity of audience; although a student's response might be viewed by a classmate or teacher, their explicit goal is to articulate their own sense-making and rough-draft thinking rather than trying to convince someone else. This rhetorical move orients the work of producing an argument as a response to a question rather than a means of completing a problem or an exercise. So, whom are students responding to when answering these questions? Implicitly there appears to be a dialogue between the textbook and the student, where the textbook issues directives ("Find every….") as well as guidance for how the student is to reflect upon their work ("Why do you think…."). In addition, as mentioned above, pressing students to explain why something is true upholds a core value held by practitioners of mathematics, but the focus on a student's own sense-making codes a lower expectation of "rigor" than would be expected in a proof.

By the third lesson of the unit, students are ready to explore the Pythagorean Theorem. The third lesson begins with it a guiding question: "Consider a right triangle with legs that each have a length of one. Suppose you draw squares on the hypotenuse and legs of the triangle. How are the areas of these three squares related?" (Lappan et al., 1997, p. 27; see Figure 3 for a visualization of this directive). Whereas the previous lesson focused on thinking about side lengths and areas of all kinds of triangles, this lesson is focusing students' attention on the relationship between the side lengths of right triangles. The first step is to have students construct a table comparing the lengths of legs with the areas of the squares constructed from the legs for right triangles of different sizes; then students are given a directive to generalize: "Use the pattern you discover to make a conjecture about the relationship among the areas" (Lappan et al., 1997, p. 28).

Up to this point in the sequence, the tasks have been preparing students to generate a conjecture and generalize what they will learn is the Pythagorean Theorem. The prompts provided for the exercises in each section have been enculturating them to the norms of knowledge building in mathematics; mathematicians explore cases, notice patterns, and then provide justifications to show that those patterns are always true. These investigations crescendo to the next investigation titled "Puzzling through a Proof" (Lappan et al., 1997, p. 29). The investigation begins by giving historical context to the "famous" theorem students will learn: the Pythagorean Theorem. Students are told that "a theorem is a general mathematical statement that has been proven true" and "over 300 different proofs have been written for this theorem." Hence, students begin this exploration knowing that the relationship that they have noticed is a known mathematical fact, and many, many proofs already exist to show that it is a fact.

The "Puzzling through a Proof" lesson is divided into four tasks (Lappan et al., 1997, p. 29):

A. How do the side lengths of the squares compare to side lengths of the triangle?
B. Fit the 11 pieces into the two frames (Figure 5).
C. What conclusion can you draw about the relationship among the areas of the three square puzzle pieces?
D. What does the conclusion you reached in part C mean in terms of the side lengths of the triangles? State this relationship as a general rule for any right triangle with legs of lengths a and b and a hypotenuse of length c.

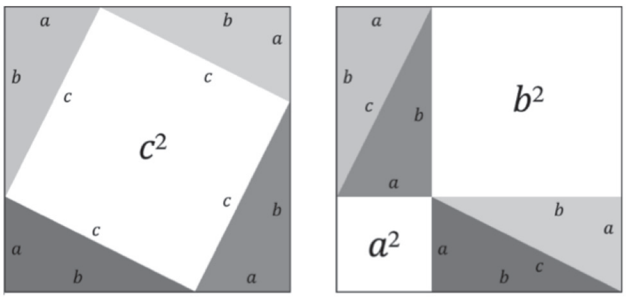


Figure 5. A visual proof of the Pythagorean Theorem (Faulk, 2014).

We would like to highlight some key aspects of this part of the task sequence. First, the framing of the activity as a puzzle may suggest to students that there is only one way to show the theorem is always true, as jigsaw puzzle pieces only fit together in one way. Second, the puzzle gives a physical confirmation of the relationship $a^2+b^2=c^2$. Mathematics educators have named this a generic example (Balacheff, 1988), as no units are shown and we could assume a, b, and c to be any numbers. This observation, if left implicit, could support problematic notions of the nature of mathematical proof. It is plausible that some students might complete this activity and take away that the Pythagorean Theorem holds for this case, with these pieces, rather than serving as a generic prototype for any right triangle. Furthermore, as an argument type, the "puzzle" shows that the relationship holds, not why it is true and, despite the heading "Puzzling through a Proof," students are not asked to explicitly connect the construction of the puzzle to an argument for why the rule is true. In terms of the expectations for arguments as specified in the ELA CCSS, the task prompt in this investigation for the CMP 1 sequence on the Pythagorean Theorem does not ask students to articulate why the completed puzzle is evidence that the theorem is true. Part D asks them to generate a "rule," so students may conclude the construction of the puzzle is to help them articulate the claim, or "rule," rather than a visual way to show why the rule is true.

Before we turn our attention to analyzing how this task sequence evolved in the third edition of CMP, we want to highlight some takeaways from our analysis of the CMP 1 sequence. In this version of the sequence, students are mathematically prepared to make a generalization, or construct a claim; students explore the connection between the side length of a triangle and the area of a square constructed from the side length of the triangle. This reinforces the concept behind the relationship between the lengths of the shorter sides and the hypotenuse of a right triangle that might otherwise not be apparent if students only examine patterns between values in a table and are then asked to generalize. Second, students have opportunities prior to thinking about a proof of the Pythagorean Theorem to justify their claims by responding to questions, rather than directives to produce an argument. For example, before they consider a proof of the Pythagorean Theorem, they are invited to think about sets of possible cases to which a relationship could apply and generate a response to a question prompting them to explain why a pattern holds for all such cases. We claim that responding to a question, rather than producing what is asked in a directive, leaves open the possibility for a response that explains rather than one that demonstrates, or shows that something is true. Finally, in the capstone activity of the sequence, the task prompts fall short of asking students to explain how their actions in producing a proof justify the Pythagorean Theorem is true for any right triangle. In the next section, we will see if the revised version of the task sequence extends the opportunities students have to generate an argument.

CMP 3: Puzzling Toward a Proof

After the publication of CMP 1, one of the significant changes to the policy landscape in mathematics education was the publication of the Common Core State Standards for Mathematics (CCSS-M, CCSSI, 2010). The CCSS includes constructing and critiquing arguments as a Standard for Mathematical Practice. Unlike the ELA CCSS, which has a number of literacy standards that discusses how students should construct viable arguments, the Math CCSS only discusses the criteria for arguments in a few sentences within the paragraph describing Standard for Mathematical Practice 3 (SMP 3):

> They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (p. 7)

One difference between the specifications of the SMP 3 and the ELA CCSS argumentation standards is the extent to which components of producing an argument are described. The description for SMP 3 provides suggestions for what students should have in their arguments (e.g., correct logic or reasoning, concrete referents), but leaves open a wide range of possibilities for the kind of arguments students can construct. The specificity of the ELA CCSS standards for argumentation as compared to SMP 3 is surprising, given the status of formal argument in building knowledge in the discipline of mathematics.

At the beginning of the CMP 3 unit, the stated Mathematical Highlights are largely similar to those originally written in CMP 1, with the addition of guiding questions including, "What are the quantities in this problem?" and "Is the Pythagorean Theorem useful and appropriate in this situation? How do I know?" and "How are the side length and the area of a square related?" (Lappan et al., 2014, p. 4). In the section following the Mathematical Highlights of the unit, the authors discuss the CCSSM-M Standards for Mathematical Practice and "habits of mind" that students will explore with the lessons in the unit:

> MP 1: Make sense of problems and persevere in solving them
> MP 2: Reason abstractly and quantitatively
> MP 3: Construct viable arguments and critique the reasoning of others (p. 5)

The text goes further to provide some guidance about how students could engage in MP 3: "When you are asked to explain why a conjecture is correct, you can: show some examples that fit the claim and explain why they fit; show how a new result follows logically from known facts and principles…" (Lappan et al., 2014, p. 5). However, consider the differences in the types of arguments students would produce following these recommendations. On one hand, showing some examples and explaining why the examples "fit" the claim is, like the puzzle proof in Figure 5, an argument of the genre of showing that a claim is true—and, precisely, that we know it is only sometimes true (for the examples shown).

Another similarity between the versions is that students begin the unit with explorations of finding lengths and areas of segments and shapes drawn on a dot grid. We see some of the same homework questions ("Find the areas of triangles AST, BST, CST, and DST. How do the areas compare? Why do you think this is true?" [Lappan et al., 1997, p. 23]), as well as added new signposts of work on Mathematical Practices: “As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems….” (Lappan et al., 2014, p. 21).

The most striking change between the two versions is the opportunities for students to reason about the Pythagorean Theorem. Unlike CMP 1, the first set of tasks in CMP 3 to explore the Pythagorean Theorem asks students to generate a table of side lengths, and areas of squares drawn from the sides, for right triangles as well as acute and obtuse triangles. They are then asked to “Make a conjecture about the areas of the squares drawn on the sides of a triangle and the type of triangle” (Lappan et al., 2014, p. 40). Like CMP 1, students are invited to make a claim about the relationship between the areas of the squares drawn on the sides of the triangle and the lesson proceeds, as in CMP 1, to name the Pythagorean Theorem and explain what a theorem is in the discipline of mathematics. The text also shares that over 300 proofs exist for the Pythagorean Theorem. However, one major shift between the versions happens after students complete the first three tasks under the heading “Puzzling through a Proof” (Lappan et al., 1997, p. 29):

A. How do the side lengths of the squares compare to side lengths of the triangle?
B. Fit the 11 pieces into the two frames.
C. What conclusion can you draw about the relationship among the areas of the three square puzzle pieces?

Instead of part D as stated in CMP 1: “What does the conclusion you reached in part C mean in terms of the side lengths of the triangles? State this relationship as a general rule for any right triangle with legs of lengths a and b and a hypotenuse of length c,” the CMP 3 version directly prompts students to reason about the truth of the statement for the general case: “Compare your results with those of another group. Did that group come to the same conclusion your group did? Is this conclusion **true for all** right triangles? Explain” (bolding added for emphasis). The CMP 3 version explicitly prompts students to explain why their conclusion is true for all right triangles. We see here, once again, the authors evoking the importance of generality in students’ arguments (“for all”). But, more importantly, we see a push beyond just stating a claim.

One unique feature of SMP 3 (CCSS, 2010) is the call for students to be able to evaluate others’ arguments. CMP 3 includes more opportunities for students to evaluate hypothetical student arguments in the textbook compared to CMP 1, and each of these cases serves as an opportunity to develop students’ understanding of the genre of mathematical argumentation. For example, as the lesson in CMP 3 transitions to a lesson on a technique for measuring using lengths that form a right triangle, students are invited to consider a sample piece of student reasoning:

> Raeka claims that if the lengths of the three sides of a triangle satisfy the relationship, $a^2+b^2=c^2$, then the triangle is a right triangle. She reasons as follows: Take the two shorter side lengths a and b. Use these to form a right angle and then a right triangle. Call the length of the hypotenuse d. Since this is a right triangle, then $a^2+b^2=d^2$. You also know that $a^2+b^2=c^2$. Therefore, $c^2=d^2$ so c=d. Since three sides of one triangle are the same as the three sides of another triangle, then these two triangles are the same. This means that the original right triangle is unique. Does Raeka’s reasoning prove the conjecture that if $a^2+b^2=c^2$, then the triangle with side lengths a, b, and c is a right triangle? Explain. (Lappan et al., 2014, p. 48)

The genre of argument here has a key difference with the “puzzle proof” in that Raeka does not generate a concrete example but uses letters to generate algebraic equations that express relationships between variables (i.e., a quantity that can vary). Yet, it also uses a mode of argumentation that differs from the modes suggested at the beginning of the unit. While the use of algebraic equations differs from showing the relationship holds for a set of examples, the argument also does not use logical statements presented deductively based on known facts and assumptions. Instead, the genre of Raeka’s argument is known in the discipline of mathematics as a “proof by construction.” Raeka constructs a right triangle that satisfies the Pythagorean Theorem, and then argues that the hypotenuse of the new triangle is equal to the hypotenuse of a triangle that satisfies the Pythagorean Theorem. Thus, if the side lengths are the all the same, it is, in fact, a unique triangle that satisfies the criteria of being a right triangle with side lengths such that squares drawn from those side lengths have a numerical relationship expressed by the Pythagorean Theorem, and not some other kind of triangle. Like rhetorical devices, different modes of mathematical argumentation may be employed based on the kind of claim being made. Modes like generic example proofs and proof by construction adhere to standards of mathematical argumentation—such as removing doubt about the truth of an assertion by building arguments based on true statements organized in a coherent and logical manner. The inclusion of this argument evaluation activity in the CMP 3 sequence enhances students' opportunities to learn different modes of argumentation than the previous sequence in CMP 1.

Summary

In this section, we have presented analyses of two task sequences designed to engage students in a central mathematical practice: generalizing and creating arguments to explain why generalizations are true. Proof is the central epistemic practice of mathematics, and these task sequences are most middle school students' first exposure to proof. Our analysis of these task sequences, specifically highlighting changes between the first and third published versions of them, reveals specific ways the textbook is communicating to students both the social value of generalization and the social value of establishing truth in the discipline of mathematics.

In both the first and third versions of the sequence, we see directives for students in terms of finding "every possible" case, to "make a conjecture," and to "find a general rule." These directives reveal that doing mathematics entails not just solving different problems and seeing if you can get the answers—doing mathematics is the practice of seeing patterns in cases you have explored. Invitations to generalize are presented as questions, although the audience listening to the response is not explicitly stated. More of the prompts invite students to consider, "How do the areas compare?" and "How do the side lengths of the squares compare to the side lengths of the triangle?" rather than "State a rule describing the relationship between the side lengths of the squares and the side lengths of the triangle." Although the difference in outcome is subtle, the difference for students is that they are invited to generalize through the process of noticing, considering, and wondering rather than completing a request from the textbook. And, when there are instances of generalizations prompted by directives (as in part D of the "Puzzling through a Proof" task from CMP 1; Lappan et al., 1997, p. 29), the opportunities presented in the textbook fail to reach a level of establishing truth of the generalization.

Between the two task sequences, there are evident differences in how each sequence communicated to students the social value of establishing truth through the task prompts. While part D from the "Puzzling through a Proof" investigation falls short in the CMP 1 version of asking students to explain why their complete puzzle illustrates that the Pythagorean Theorem is true for all right triangles, the third version from CMP 3 invites students to explain why the conclusion they drew from their work on the puzzle is true for all triangles. Moreover, part D asks students to do this as a result of their discussions with another group, not just as a result of their personal explorations. The modifications to the prompt suggest critical aspects relevant to the authenticity of students' engagement in the disciplinary practice of proof. First, authoring mathematics does not happen in isolation but, rather, as a community of practice. Second, it is not enough to provide a case that shows that a statement is true: Doing mathematics entails explaining that the statement is true for all cases.

Conclusion

Our critical examination of these sequences raises questions about engaging students in epistemic practices like proof that an audience with expertise in developing students' literacy might be particularly poised to address. First, we noted aspects related to audience in the prompts in the task sequence. If we wish students to approach these task sequences from a frame of mind of authoring mathematical ideas, rather than more passive stances such as practicing skills or experiencing mathematics, how might the way we word directives or pose questions in the text achieve this frame of mind? Should the prompts leave the audience undefined? Could directives to prepare an argument to share with the teacher and/or classmates prompt students to better perceive the epistemic role of argumentation in mathematics?

Second, we are curious about the use of certain puzzles as a context for exploring mathematical proof with regard to the Pythagorean Theorem. The completion of the puzzle is a form of argumentation known as a generic example (Balacheff, 1988) in mathematics, but some would argue that it is not a proof without the companion explanation of how the puzzle reflects behavior that could be generalized to all right triangles. Given that middle school students are just beginning formal study of algebra and Euclidean geometry, other forms of argumentation to show why the Pythagorean Theorem holds true for all cases is beyond their reach. And herein lies the tension of the work around incorporating more opportunities to learn about the important role of proof in mathematics in school mathematics, making the work of creating a proof accessible for students. Even when students have the apparatus to make formal arguments, as they learn in high school geometry, teachers often describe the process of teaching students to prove as throwing them into the deep end of a pool (Cirillo, 2008). The field of mathematics education would benefit from learning how to orchestrate communicating disciplinary standards and providing students with authentic experiences as authors of mathematics appropriate to their level of understanding. Doing this throughout the K–12 curriculum remains a formidable challenge.

In this article, we discussed the nature of the key tool in mathematics for establishing knowledge—argumentation—and the aspects of the genre of argumentation that are important norms for the discipline. We looked at how these aspects emerge in task sequences for middle school students when learning about the Pythagorean Theorem, and raised questions about what students may derive about the mathematical practice of argumentation from these experiences. We hope that we have provided some clarity around what argumentation is in mathematics and, more importantly, what mathematics teachers are trying to help students do with regards to engaging in argumentation that honors its role in the discipline. We invite you to have conversations with mathematics teachers about best practices for positioning students to respond to these kinds of tasks in mathematics textbooks so that they experience argumentation as an epistemic practice and not as another exercise to complete.

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