Jacqueline Stewart and Elizabeth Phillips, Connected Mathematics Project, Michigan State University

”Making Sense of Symbols, Writing Equivalent Expressions: *Say It with Symbols* Problem 1.1

LAUNCH:

Before viewing the video of students doing *Say It With Symbols*, 1.1.

Getting Ready to View the Video: Do 1.1

Before viewing the video participants need to do *Say It With Symbols*, 1.1, just as if they were planning to teach the lesson. I think I will launch 1.1 by having a “teacher” discussion about what *equivalence* means, what role it plays in the study of algebra, and what mathematical experiences one would expect CMP students to draw on for 1.1. Then we will do Inv 1.1 in its entirety, Launch- Explore and Summarize.

After the “student” summary of Problem1.1 I can select a few “teacher” questions from the list below, to help participants prepare their mindsets to watch the video of students doing the same problem.

After participants have done 1.1	**Possible “Teacher” Discussion Questions:**	**In Previous Workshops Teachers Have Said:**	**Follow Up Questions**
Questions for Teachers	• What expressions do you predict your students will generate?	- *What if they don’t generate many expressions?* - $n = 4s + 4$, $n = 4(s + 1)$, $n = 2s + 2(s + 2)$	- Should we offer expressions from “another class?” - Are the expected expressions connected?
Note: the first two questions are always part of planning to teach.	• What difficulties might you predict for your students in 1.1?	- *Maybe the picture of a square pool will make them think of area, not perimeter.* - *Some may not be able to write any general rule in symbols.*	- Is the number of border tiles related to perimeter? Area? - How can we help students get started to thinking about a general rule? [Ω]

[Ω] Some students find it difficult to write symbolic expressions. Verbalizing the steps in each student computation as the result is entered in a table helps to generalize any process more sophisticated than just counting. For example, they may count a side plus a corner plus a side plus a corner, and then double all that. This generalizes as $(s + 1 + s + 1)•2$. The teacher can record the student -verbalized computation in words and symbols as a preparatory step en route to having students do this independently. Trying to make contextual sense of expressions written by others re-inforces this. Some students may also be helped by counting the perimeters of specific examples and recording these in a table. For example, if a student draws a 3x3, 4x4 and 5x5 pool and counts the number of perimeter tiles, then they have no access to a strategy that generalizes; but they will still produce a table of results that suggests a linear equation with a “4x” term.

	Possible "Teacher" Discussion Questions (cont'd):	**In Previous Workshops Teachers Have Said:**	**Follow Up Questions**
Note: Try to keep the focus on the mathematics.	• What do you think students will understand about *equivalence* prior to 1.1? Where did they get these understandings?	- *They have used this word in relation to fractions, ratios, and expressions. They may also have used it in Moving Straight Ahead or Growing, Growing, in relation to alternative equations.*	- In FF students have used the distributive property to multiply two binomials, producing an equivalent expression. Is this relevant to this problem?[Ω]
	• What mathematical ideas would you want to come out of Problem 1.1?	- *Students connect the context to a symbolic expression* - *Students use tables and graphs to check equivalence* - *Students begin to connect symbolic expressions* - *Hope they see that the number of tiles grows linearly.*	- What should students already know about properties of real numbers that will allow them to manipulate symbols? - Should we push these manipulation skills if they don't come up?[Ω]

[Ω] It's interesting to me that the teacher on the video used the example of multiplying 2 binomials to remind students of their knowledge about equivalent expressions, yet, as we shall see on the video, almost no students tried to use the area of the pool to work out the number of border tiles. One student, Heather, specifically mentions that multiplying x•x will give the area of the pool, but she does this as a way of explaining that that is *not* what she wants or needs. Another student, Ellie, has real trouble coming up with even one expression. For example, she suggests "$4x^2$" in an effort to reconcile the 4 sides she sees and the square she sees. She finally settles on x•4 + 4, but when she writes her expression on the board she writes x•4x + 4. Is this a slip or is Ellie still thinking she has to multiply the two sides somehow? It's not possible to tell without asking Ellie.

[Ω] Problems 1.3 and 1.4 will focus more on symbolic manipulation, specifically the Distributive Property. Listening to students carefully, the teacher will have to follow their lead at this stage, knowing there are other opportunities coming up.

	Possible "Teacher" Discussion Questions (cont'd):	**In Previous Workshops Teachers Have Said:**	**Follow Up Questions**
	• What will your role be in the Summary?	- *Choose different strategies and push students to validate their expressions by reasoning about the geometric context* - *Probe ways to check equivalence* - *Ask about linearity*	- Would you have selected different "student" work from what I did? - Would you have sequenced or connected this differently? - Do similar graphs prove relationships are equivalent?
	• How does this problem, by itself, help students make sense of symbols?	- *They have to write an expression and be able to make sense of others' expressions.* - *They have to use order and parentheses carefully.* - *They have to connect their expression to the context or to their computation.*	- In a single expression $n = 4 \cdot (x + 1)$ there are 9 symbols. Can we change the order of the symbols, say $n = (x \cdot 4) + 1$? Does the context help students make sense of *order*?
Questions for Curriculum leaders (and teachers)	• How does this problem advance the idea of *equivalence*?	- *In FF students used tables and graphs and distributive property, but all related to quadratic forms only. The solutions are all linear here.* - *This time there are many equivalent forms.* - *We get different information from each form; one form is not considered best.*	- What do 8th grade teachers need to know about *equivalence*? How can they increase their knowledge?

Possible "Teacher" Discussion Questions (cont'd):	**In Previous Workshops Teachers Have Said:**	**Follow Up Questions**
• Is *equivalence* a big idea on your State Standards?	-	- Name some State Standards statements, or some examples of test items, that imply an understanding of *equivalence*.

VIDEO: "Making Sense of Symbols: Writing Equivalent Expressions (*Say It With Symbols*, 1.1)"

27 minutes, 17 chapters

Note: This video has been edited to focus on students making sense of symbols, and the teacher's role in setting up an environment where making sense is the norm. Real time was 1.5 class periods.

EXPLORE: While Watching the Video

The following focus questions help both teachers and curriculum leaders think about where students are in their development of symbol sense. Each person (or group) should select one or two questions to focus on while watching the video.

Focus Questions

- What evidence do we see of students making sense of symbolic expressions? How do they make sense of symbols?
- What evidence is there that students understand *equivalence*? What do they still have to understand about *equivalence*?
- What do students understand about linear expressions? Quadratic expressions? Is this knowledge helpful in making sense of symbols?
- What evidence is there that some students are concentrating on links among symbolic expressions rather than linking each expression to some other representation?
- What moments seem to be mathematically significant in terms of the idea of *equivalence*?
- What evidence is there that students expect to make sense? What role does the teacher play in this expectation?
- What evidence is there that the teacher purposefully chose and sequenced particular student expressions? Would you have done this differently?
- Did the teacher have to deal with unexpected student questions or comments? What came out of these impromptu situations? How would you have dealt with these?

Note:
Most of these questions focus on mathematics. An alternative to this is to focus only on Discourse, using a rubric such as the one developed by K.Huffer-Ackles et.al. (2004). See Appendix.

Focusing on Discourse *can* have the effect of relegating the mathematics to a subordinate position. See "Student Discourse: Say It With Symbols 1.3."

Form Focus Groups of Teachers

It has worked well in the past to re-arrange participants into focus groups before viewing the video. If they have a few minutes to talk about the focus question *before* the video and then time to debrief in small groups *after* the video I have noticed that the discussions are more coherent. I have tried to think of follow up questions that will help participants extend their thinking. I should keep notes from each professional development workshop so I can refine these questions.

SUMMARIZE: Focus group Discussion after viewing the video	**Focus Questions (as above)**	**In Previous Workshops Teachers have said**	**Follow Up Questions**
Note: Alternative ways to conduct discussions: It can be unnecessarily repetitive if the same discuss/ view/discuss format is followed in every pd session. I have tried different formats. Some of these are described in the appendix.	• What evidence do we see of students making sense of symbolic expressions? How do they make sense of symbols?	- *They are attaching each expression to the geometric context. If it makes sense in the context then it's correct* - *They make a table and graph—if the table or graph matches another that makes sense then it's correct.*	- How does student understanding of symbolic expressions connect to or extend what we saw them doing in *MSA* or in *Growing, Growing* or *FF*? - What are some obstacles to making sense of symbols in this problem?[Ω]
	• What evidence is there that students understand *equivalence*? What do they still have to understand about *equivalence*?	- *The evidence shows they understand that equivalent means "another way of describing" a relationship* - *They also understand that equivalence means "has same table or graph."* - *It's not clear that they all realize that the symbols can be manipulated independent of the context.*	- Can you think of instances where deliberately writing some algebraic expression in a different but equivalent way would be of practical help in solving a problem? - What are some drawbacks of the table/graph way of checking equivalence?[Ω]

[Ω] The drawing of the pool may suggest an area computation, which may lead students to expect a quadratic expression. See footnotes on pages 3 and 8.

[Ω] Two graphs may look identical, but not actually represent the same underlying relationship. See "Making Sense of Symbols: Exponential Decay (*Growing, Growing* 4.1)"

Focus Questions cont'd (as above)	**In Previous Workshops Teachers have said**	**Follow Up Questions**
• What do students understand about linear expressions? Quadratic expressions? Is this knowledge helpful in making sense of symbols?	- *They recognize linear graphs.* - *The table of a linear relation should show a constant rate of increase.* - *They know the simplest form of a linear equation looks like $y = mx + b$* - *They know that a quadratic has a term with x•x. so Ellie's equation was wrong.*	- It looks like they are beginning to pay attention to the overall form of an equation. (linear, quadratic) Why would this be helpful?[Ω] - How does this problem resemble the problems in Frogs and Fleas and how is it different?
• What evidence is there that some students are concentrating on links among symbolic expressions?	- *One student said that "(2x)2" was the same as "4x."* - *John came up with "an infinite amount" of expressions. He explicitly said he didn't have to draw a model.*	- No student came up with $n = (s + 2)^2 – s^2$. How does this expression relate to finding the number of border tiles? Would students be able to show it is equivalent to $n = 4x + b$? [Ω]

[Ω] See discussion of $n = x \cdot 4x + 4$ on page 3

[Ω] One equivalent equation that is clearly not in simplest form is: $Y = (x + 2)^2 – x^2$. This looks quadratic. Sometimes students propose this equation, because they focus on the area of the border. If no student proposes this should the teacher raise this idea? In this case, there is the possibility of an interesting discussion about whether this proposed expression is indeed quadratic. Students can apply their knowledge about the Distributive Property, gained from *Frogs and Fleas,* to write this in expanded form. In addition, asking how this expression relates to the context might stimulate a conversation about whether the original question is about the *area* of the border tiles, or the *number* of the border tiles, and whether this distinction matters. In fact, the number of border tiles in this case is a number of square feet, and thus, implicitly an area.

Focus Questions cont'd (as above)	**In Previous Workshops Teachers have said**	**Follow Up Questions**
• What moments seem to be mathematically significant in terms of the idea of *equivalence*?	- *John's manipulation of one form to get another—can't draw picture but doesn't think he has to* - *Ellie asking if the order of operations matters* - *Josh saying that the second equation is just "4x + 4"* - *A student saying that (2x)2 + 4 is just "splitting the 4x"*	- What mathematical ideas do students have to understand in order to be able to check equivalence symbolically?
• What evidence is there that students expect to make sense? What role does the teacher play in this expectation?	- *It looks like the class norm is to explain. Audrey volunteered to do this.* - *Heather disagreed with an expression, without the teacher indicating it was wrong.* - *Josh and Audrey gave additional reasons for "x4x + 4" being wrong.* - *The teacher asks why solutions make sense, and she waits for students to offer corrections.*	- *How does one set up this kind of classroom participation structure? Did Kathy do anything in setting up the summary procedure that supported this?* (See Hufferd-Ackles, 2004, in the References., to find vocabulary to talk about setting up a "math-talk community.")

Focus Questions cont'd (as above)	**In Previous Workshops Teachers have said**	**Follow Up Questions**
• Did the teacher have to deal with unexpected student questions or comments? What came out of these impromptu situations? How would you have dealt with these? Can you plan for these?		- *How did Kathy deal with the erroneous n= x•4x + 4? Do you think she expected this? What came out of this situation?*[Ω] - *How did Kathy deal with John's comment about "an infinite amount?" How might comments like this affect the teacher's plans?*[Ω]

[Ω] I don't think that Kathy expected this error, *n = x•4x + 4*. In fact I *think* she selected Ellie to validate the effort Ellie had put into finding an expression that was correct, at least at the moment that Kathy saw her write it in her notebook. Or perhaps Kathy saw n = 4x + 4 in several notebooks and wanted to give positive feedback to several groups right away. However, Kathy let the error stand until *students* corrected it. This took a long time to happen, but Kathy did not interrupt, though we can see on the original tape that she has noted that some students disagree with the expression almost immediately. One of the unintended consequences of allowing the error to stand is that when students explain why it is an error we get some insight into what they understand. They know that n = x•4x + 4 is wrong because it produces mismatches with the number of perimeter tiles in their drawings. But they also know that it is wrong because it is a quadratic expression, and they are sure that the relationship between number of tiles and side length is linear, based on the constant rate of increase shown in their drawings and tables. Without the error the discussion of forms and rates of increase would not have happened. This is one of those moments which a teacher can not plan for, but can prepare for, by doing the problem and thinking about all the mathematics in the problem and what students will bring to the problem: the linearity of the relationship, the likely responses.

[Ω] John's comment and the comments of others about symbol manipulation seems to lead Kathy to ask students if they can show the different forms are equivalent to each other or to n = 4x + 4. The Summary of 1.1 included discussion which might otherwise not have come up until Problem 1.2. The teacher chose not to do 1.2, perhaps because of the rich discussion prompted by John's and other students' comments.